

Sparsifying Eulerian Graphs

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EaGL 2024 Workshop
[\[SODA'25\]](#)

Joint with



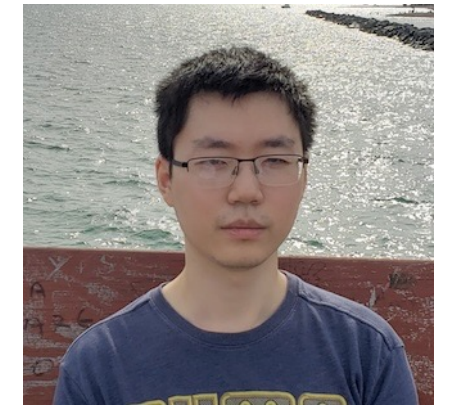
Arun
Jambulapati



Aaron
Sidford



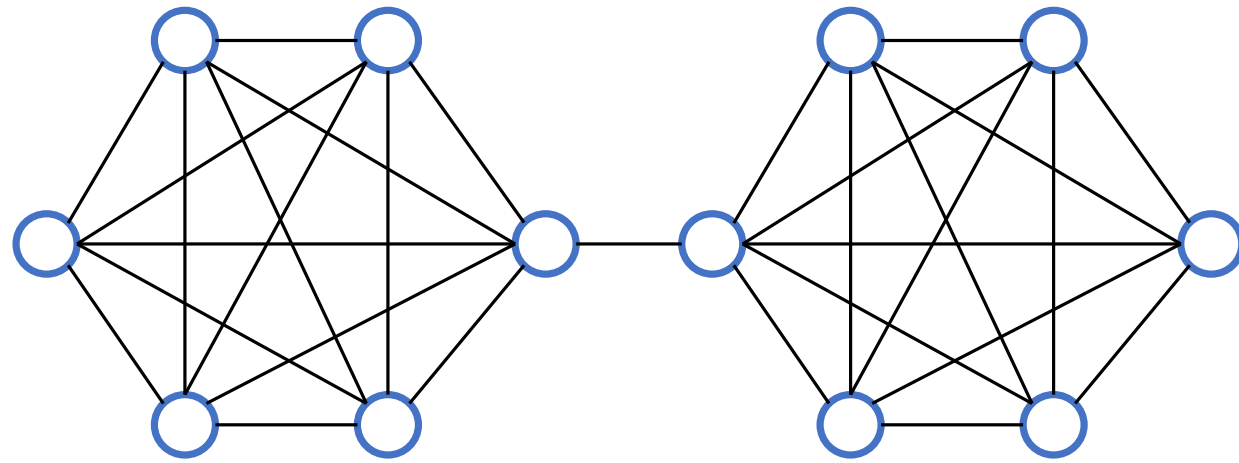
Kevin
Tian



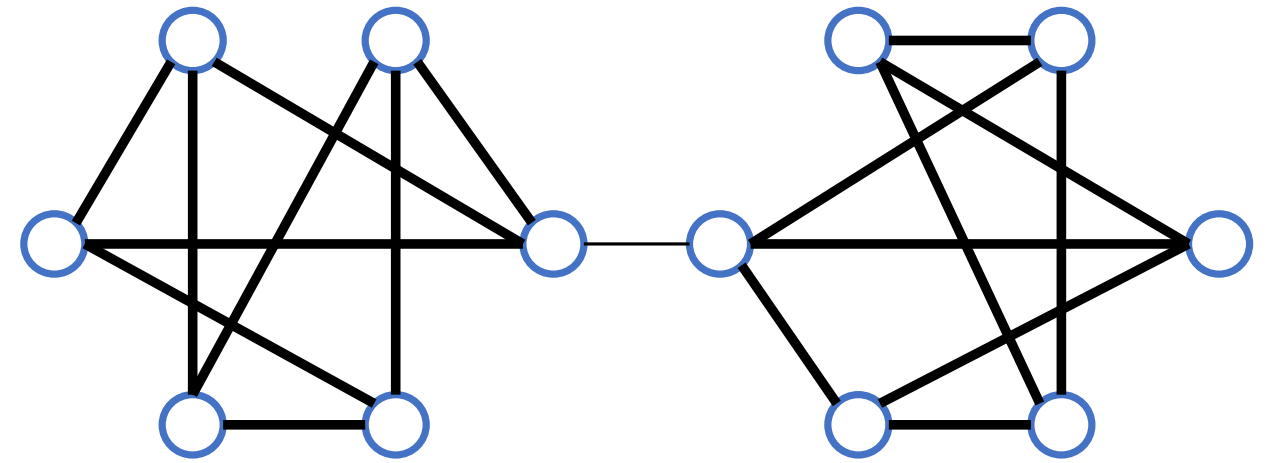
Yibin
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Sparsification

Graph Sparsifiers

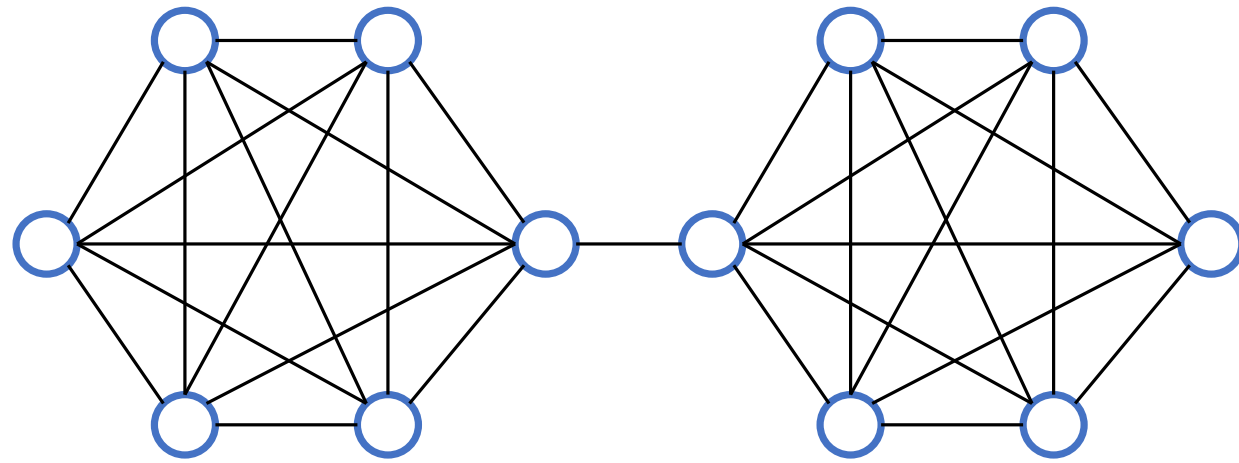


G

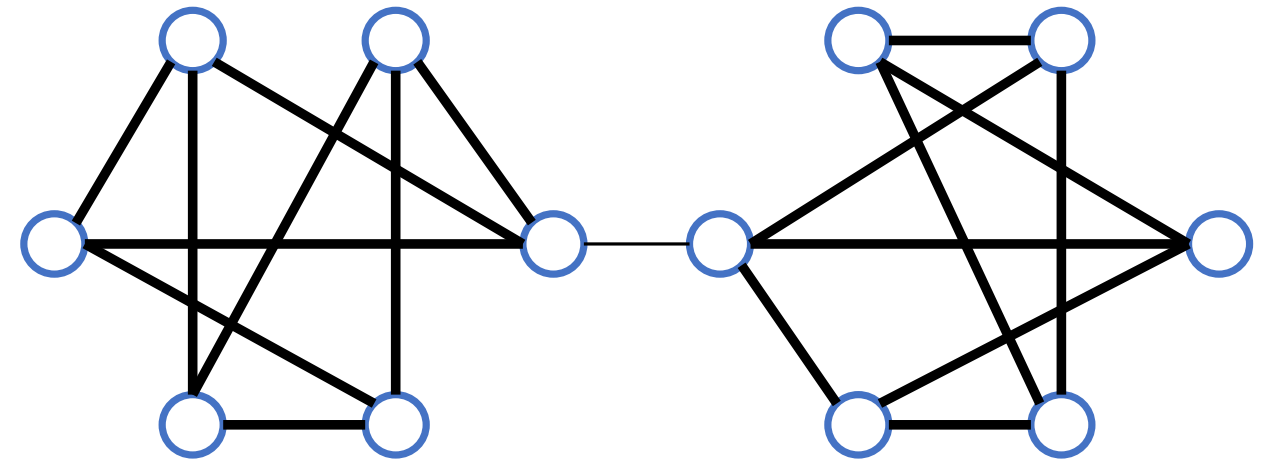


H

Spanners



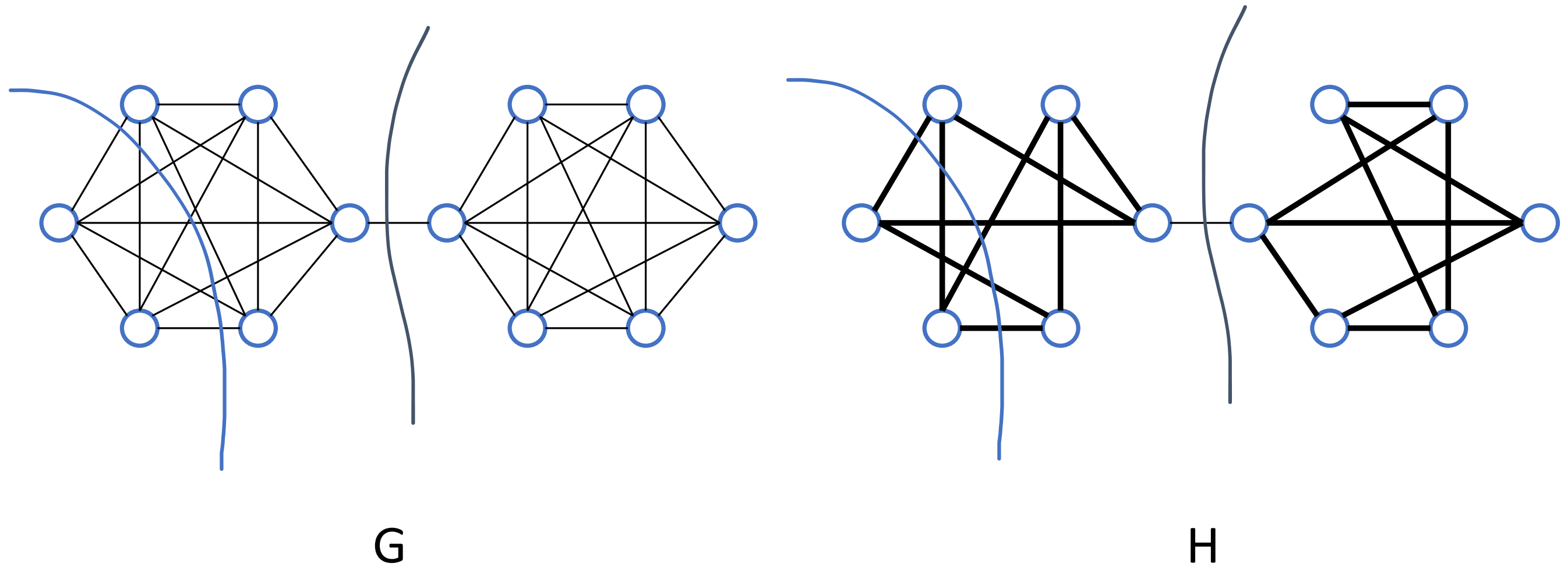
G



H

Preserve approximate shortest path distances [Chew'86]

Undirected Cut Sparsifiers



All vertex cuts have approximately equal weights in G and H

[Benczur-Karger '92] There are undirected cut sparsifiers with $n \log n$ edges

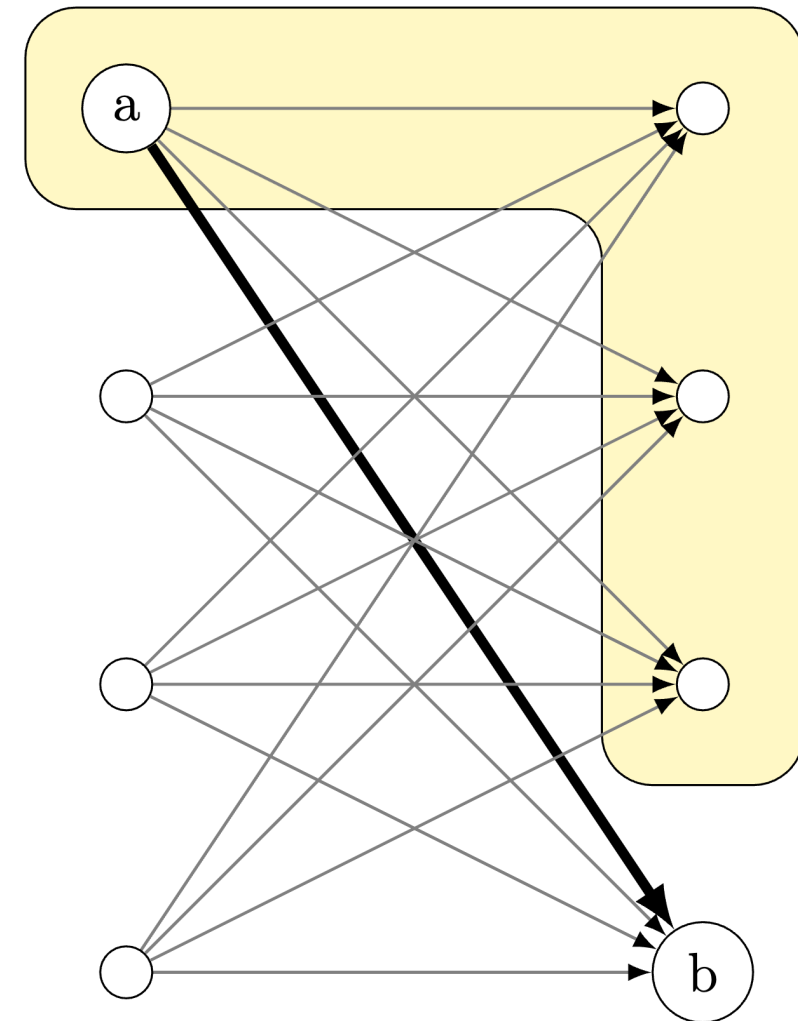
Directed Cut Sparsifiers?

Complete bipartite graph

The only outgoing edge is $a \rightarrow b$

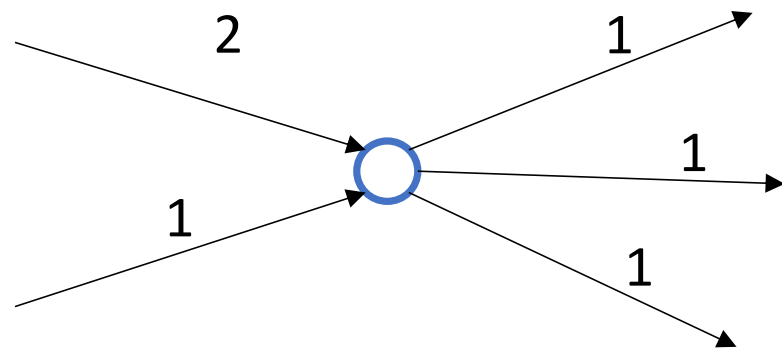
Every edge must be preserved for a multiplicative cut approximation

No sparsification possible



Eulerian (Directed) Graphs

Eulerian iff weighted in-degree = weighted out-degree



Avoids pathological issues

Observed advantage for flow problems [Ene-Miller-Pachocki-Sidford'16]

Spectral Characterization

Undirected Laplacians

A weighted, **undirected** graph $G = (V, E, w)$

$$n = |V|, \quad m = |E|, \quad w: E \rightarrow \mathbb{R}_+$$

Laplacian of edge uv is the symmetric $n \times n$ matrix,

$$L_{uv} = \begin{matrix} & \begin{matrix} u & & v \end{matrix} \\ \begin{pmatrix} 1 & \cdots & -1 \\ \vdots & \ddots & \vdots \\ -1 & \cdots & 1 \end{pmatrix} & \begin{matrix} u \\ v \end{matrix} \end{matrix}$$
$$= (\chi_u - \chi_v)(\chi_u - \chi_v)^\top$$

Undirected Laplacians

A weighted, **undirected** graph $G = (V, E, w)$

$$n = |V|, \quad m = |E|, \quad w: E \rightarrow \mathbb{R}_+$$

Laplacian of G is

$$\begin{aligned} L &= \sum_{(u,v) \in E} w_{uv} L_{uv} \\ &= \sum_{(u,v) \in E} w_{uv} (\chi_u - \chi_v)(\chi_u - \chi_v)^\top \end{aligned}$$

Directed Laplacians

A weighted, **directed** graph $G = (V, \vec{E}, w)$

$$n = |V|, \quad m = |\vec{E}|, \quad w: \vec{E} \rightarrow \mathbb{R}_+$$

Laplacian of edge $u \rightarrow v$ is the **asymmetric** $n \times n$ matrix,

$$\vec{L}_{u \rightarrow v} = \begin{matrix} & \begin{matrix} u & & v \end{matrix} \\ \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ -1 & \cdots & 0 \end{pmatrix} & \begin{matrix} u \\ v \end{matrix} \end{matrix}$$
$$= (\chi_u - \chi_v)\chi_u^\top$$

Directed Laplacians

A weighted, **directed** graph $G = (V, \vec{E}, w)$

$$n = |V|, \quad m = |\vec{E}|, \quad w: \vec{E} \rightarrow \mathbb{R}_+$$

Laplacian of \vec{G} is

$$\begin{aligned} \vec{L}_G &= \sum_{u \rightarrow v \in \vec{E}} w_{uv} L_{u \rightarrow v} \\ &= \sum_{u \rightarrow v \in \vec{E}} w_{uv} (\chi_u - \chi_v) \chi_u^\top \end{aligned}$$

Undirected Laplacian

$$L = \sum_{(u,v) \in E} w_{uv} \begin{pmatrix} u & & v \\ 1 & \dots & -1 \\ \vdots & \ddots & \vdots \\ -1 & \dots & 1 \\ & & v \end{pmatrix} \begin{matrix} u \\ \\ v \end{matrix}$$

Symmetric

$$x^\top L x = \sum_{(u,v) \in E} w_{uv} (x_u - x_v)^2$$

$x = \mathbf{1}_S$ gives cut value

$$L \mathbf{1} = L^\top \mathbf{1} = 0$$

Directed Laplacian

$$\vec{L} = \sum_{u \rightarrow v \in \vec{E}} w_{uv} \begin{pmatrix} u & & v \\ 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ -1 & \dots & 0 \\ & & v \end{pmatrix} \begin{matrix} u \\ \\ v \end{matrix}$$

(mostly) asymmetric

$$x^\top \vec{L} x = \sum_{u \rightarrow v \in \vec{E}} w_{uv} (x_u - x_v) x_u$$

$x = \mathbf{1}_S$ gives directed cut value

$$\vec{L}^\top \mathbf{1} = 0$$

Undirected Spectral Sparsifiers [Spielman-Teng'04]

Suppose G, H are graphs on the same vertex set.

Say $L_G \approx_\varepsilon L_H$

$$\forall x \quad |x^\top L_H x - x^\top L_G x| \leq \varepsilon \cdot x^\top L_G x$$

Picking $x = \mathbf{1}_S$,

$$w_H(E(S, S^c)) = (1 \pm \varepsilon) w_G(E(S, S^c))$$

Generalizes cut-sparsification

Undirected Spectral Sparsifiers [Spielman-Teng'04]

Suppose G, H are graphs on the same vertex set.

Say $L_G \approx_\varepsilon L_H$

$$\forall x \quad |x^\top L_H x - x^\top L_G x| \leq \varepsilon \cdot x^\top L_G x$$

[Spielman-Srivastava '08]

For every graph G , we can find a re-weighted subgraph H with $O(n\varepsilon^{-2} \log n)$ edges such that $L_G \approx_\varepsilon L_H$

Directed Spectral Sparsifiers

Quadratic form can be negative

Multiplicative approximations not meaningful

Define it only for Eulerian graphs

$$x^\top \vec{L} x = \frac{1}{2} x^\top (\vec{L} + \vec{L}^\top) x$$

Observation: $\frac{1}{2} (\vec{L} + \vec{L}^\top)$ is the Laplacian of the undirected symmetrized graph

Directed Eulerian (Spectral) Sparsifiers

[Cohen-Kelner-Peebles-Peng-Rao-Sidford-Vladu'17]

$U = \frac{1}{2} (\vec{L} + \vec{L}^\top)$ is the Laplacian of the undirected symmetrized graph

$$\forall x, y \quad |x^\top \vec{L}_H y - x^\top \vec{L}_G y| \leq \varepsilon \cdot \sqrt{x^\top U_G x \cdot y^\top U_G y}$$

Generalizes undirected spectral sparsification

For an Eulerian graph, $\vec{L} \mathbf{1} = \vec{L}^\top \mathbf{1} = 0$

Sparsifier of Eulerian directed graph must be Eulerian

Applications

[Cohen-Kelner-Peebles-Peng-Rao-Sidford-Vladu'17]

Fast algorithms for Eulerian sparsification give fast algorithms for:

- Solving directed Laplacian systems $\vec{L}_G x = b$ (not necc Eulerian)
 - Computing stationary distributions of Markov chains
 - Computing commute times
 - Personalized page rank vectors
 - Top eigenvector (Perron–Frobenius vector) for positive matrices
- [Ahmadinejad-Jambulapati-Saberi-Sidford'21]

Approaches to Eulerian Sparsifiers

How to build Eulerian sparsifiers v1

[Cohen-Kelner-Peebles-Peng-Rao-Sidford-Vladu'17]

Sample edges independently in expanders

Fix degree imbalance by adding extra edges

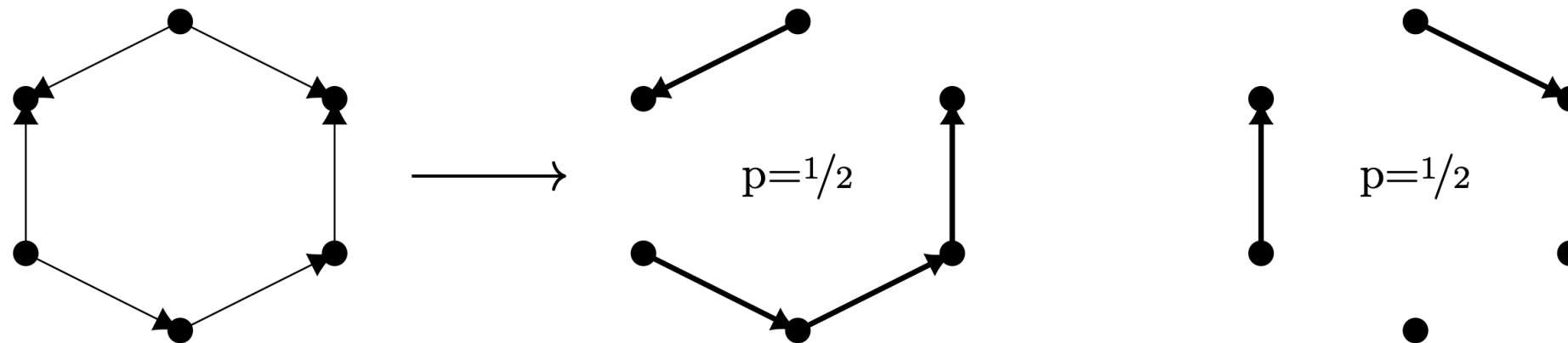
$n \log^C n$ edge sparsifier in $m \log^C n$ for large C

How to build Eulerian sparsifiers v2

Short cycle decompositions [Chu-Gao-Peng-S-Sawhani-Wang'18]

Any unit-weighted graph can be partitioned into edge-disjoint cycles of length at most $2 \log n$, with $2n$ leftover edges.

Toggle cycles independently, keep edges in one direction with double weight



Previous Best Guarantees

Previous Best Guarantees

Approach

Sparsity

Runtime

Previous Best Guarantees

Approach

Expander decompositions

[Cohen-Kelner-Peebles-Peng-Rao-Sidford-Vladu'17,
Ahmedinajad-Peebles-Pyne-Sidford-Vadhan'23]

Sparsity

$n \log^{20} n$

Runtime

$m \log^7 n$

Previous Best Guarantees

Approach

Expander decompositions

[Cohen-Kelner-Peebles-Peng-Rao-Sidford-Vladu'17,
Ahmedinajad-Peebles-Pyne-Sidford-Vadhan'23]

Short cycle decomposition

[Chu-Gao-Peng-S-Sawhani-Wang'18,
Liu-S-Yu'19, Parter-Yogev '19, Thudi-S-Zhao'24]

Sparsity

$$n \log^{20} n$$

$$n \log^3 n$$

Runtime

$$m \log^7 n$$

$$m^{1+\delta}$$

Previous Best Guarantees

Approach

Expander decompositions

[Cohen-Kelner-Peebles-Peng-Rao-Sidford-Vladu'17,
Ahmedinajad-Peebles-Pyne-Sidford-Vadhan'23]

Short cycle decomposition

[Chu-Gao-Peng-S-Sawhani-Wang'18,
Liu-S-Yu'19, Parter-Yogev '19, Thudi-S-Zhao'24]

Electrical flows + effective resistance decomposition

[Jambulapati-S-Sidford-Tian-Zhao'24]

Sparsity

$$n \log^{20} n$$

$$n \log^3 n$$

$$n \log^2 n$$

Runtime

$$m \log^7 n$$

$$m^{1+\delta}$$

$$m \log^3 n$$

Building Undirected Sparsifiers

Edge Leverage Scores

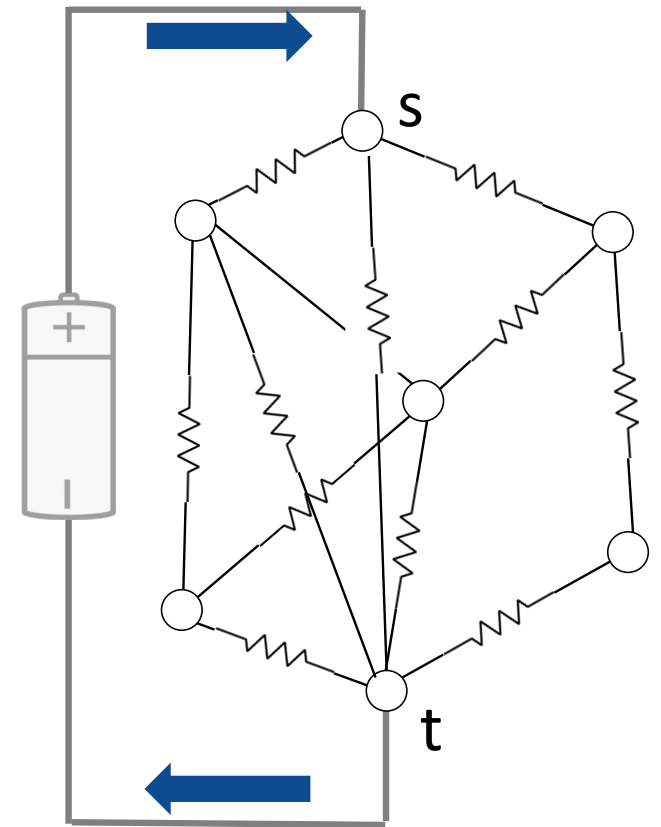
Undirected graph G ,

Define effective resistance of an edge e

$$\begin{aligned} r_e &= (\chi_u - \chi_v)^\top L^{-1} (\chi_u - \chi_v) \\ &= \text{Tr}(L_e L^{-1}) \end{aligned}$$

Define leverage score of e as $w_e r_e$

Observe: $\sum_e w_e r_e = \text{Tr}(L^{-1} \sum_e w_e L_e) \leq n$



Constructing Undirected Sparsifiers

Pick edges with leverage score $w_e r_e \leq 2n/m$

Toss independent coins for edges

Tails: Discard the edge, or,

Heads: Keep the edge and double the weight

Repeat for $\log n$ iterations

Scalar Azuma's Inequality

Consider values $X_i \in R$ s.t

$$\sum_i X_i^2 \leq \sigma^2$$

Then, for random $s_i \sim \pm 1$,

$$\left| \sum_i s_i X_i \right| \leq \sigma \sqrt{\log m}$$

with high prob.

Scalar Azuma's Inequality

Consider values $X_i \in R$ s.t

$$\text{Var} \left(\sum_i s_i X_i \right) \leq \sigma^2$$

Then, for random $s_i \sim \pm 1$,

$$\left| \sum_i s_i X_i \right| \leq \sigma \sqrt{\log m}$$

with high prob.

Matrix Azuma's Inequality

[Lust-Piquard & Pisier'91, Tropp '11]

Consider matrices $X_i \in R^{n \times n}$ s.t

$$\max \left\| \sum_i X_i X_i^\top \right\|, \left\| \sum_i X_i^\top X_i \right\| \leq \sigma^2$$

Then, for random $s_i \sim \pm 1$,

$$\left\| \sum_i s_i X_i \right\| \leq \sigma \sqrt{\log m}$$

with high prob.

Matrix Azuma's Inequality

[Lust-Piquard & Pisier'91, Tropp '11]

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$$\left\| \text{Var} \left(\sum_i s_i X_i \right) \right\| \leq \sigma^2$$

Then, for random $s_i \sim \pm 1$,

$$\left\| \sum_i s_i X_i \right\| \leq \sigma \sqrt{\log m}$$

with high prob.

Constructing Undirected Sparsifiers

Want to prove

$$-\delta \cdot x^\top L_G x \leq x^\top L_H x - x^\top L_G x \leq \delta \cdot x^\top L_G x$$

Constructing Undirected Sparsifiers

Want to prove

$$\left\| L_G^{-1/2} (L_H - L_G) L_G^{-1/2} \right\| \leq \delta$$

Constructing Undirected Sparsifiers

Want to prove

$$\left\| \sum_e \Delta w_e L_G^{-1/2} L_e L_G^{-1/2} \right\| \leq \delta$$

Suffices

$$\left\| \sum_e w_e^2 L_G^{-1/2} L_e L_G^{-1} L_e L_G^{-1/2} \right\| \leq \frac{\delta^2}{\log n}$$

Constructing Undirected Sparsifiers

Want to prove

$$\left\| \sum_e \Delta w_e L_G^{-1/2} L_e L_G^{-1/2} \right\| \leq \delta$$

Suffices

$$\left\| \sum_e w_e^2 L_G^{-1/2} \chi_e \chi_e^\top L_G^{-1} \chi_e \chi_e^\top L_G^{-1/2} \right\| \leq \frac{\delta^2}{\log n}$$

Constructing Undirected Sparsifiers

Want to prove

$$\left\| \sum_e \Delta w_e L_G^{-1/2} L_e L_G^{-1/2} \right\| \leq \delta$$

Suffices

$$\left\| \sum_e w_e^2 r_e L_G^{-1/2} L_e L_G^{-1/2} \right\| \leq \frac{\delta^2}{\log n}$$

Constructing Undirected Sparsifiers

Want to prove

$$\left\| \sum_e \Delta w_e L_G^{-1/2} L_e L_G^{-1/2} \right\| \leq \delta$$

Suffices

$$\max_e w_e r_e \cdot \left\| \sum_e w_e L_G^{-1/2} L_e L_G^{-1/2} \right\| \leq \frac{\delta^2}{\log n}$$

Constructing Undirected Sparsifiers

Want to prove

$$\left\| \sum_e \Delta w_e L_G^{-1/2} L_e L_G^{-1/2} \right\| \leq \delta$$

Suffices

$$\max_e w_e r_e \leq \frac{\delta^2}{\log n}$$

Constructing Undirected Sparsifiers

Want to prove

$$\left\| \sum_e \Delta w_e L_G^{-1/2} L_e L_G^{-1/2} \right\| \leq \delta$$

Suffices

$$\frac{2n}{m} \leq \frac{\delta^2}{\log n}$$

Constructing Undirected Sparsifiers

We obtain graph H such that $L_H \approx_\delta L_G$ for $\delta = \sqrt{\frac{n \log n}{m}}$

Total error on iterating

$$\sqrt{\frac{n \log n}{m}} + \sqrt{\frac{n \log n}{3m/4}} + \dots$$

Stop when last term is ε , for total error $O(\varepsilon)$

Giving final edge count $m' = n\varepsilon^{-2} \log n$

Building Eulerian Sparsifiers

Constructing Eulerian Sparsifiers?

Obstacle:

Pick random signing $s_e \sim \pm 1$, $\Delta w_e = s_e w_e$ need not preserve Eulerianness

Previous solutions:

Only toggle directed cycles => Expensive to find short cycles

Add extra edges => Challenging to keep error small

Constructing Eulerian Sparsifiers?

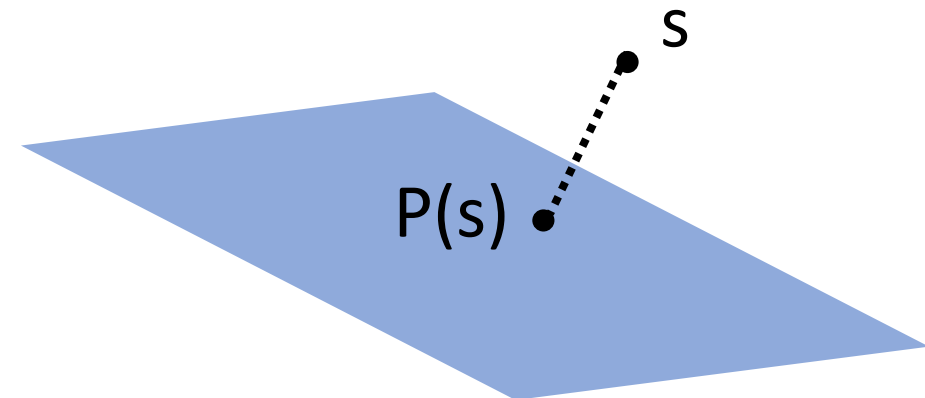
Obstacle:

Pick random signing $s_e \sim \pm 1$, $\Delta w_e = s_e w_e$ need not preserve Eulerianness

Our solution:

Pick random signing $s_e \sim \pm 1$ and linearly project s to ensure Eulerianness

Equivalent to fixing degree balance
by using electrical flows to route excess



Constructing Eulerian Sparsifiers?

Pick random signing $s_e \sim \pm 1$ and linearly project s to ensure Eulerianness

Challenge 1: What happens to the variance?

Key Observation:

$$\left\| \text{Var} \left(\sum_e P(s)_e w_e X_e \right) \right\| \leq \left\| \text{Var} \left(\sum_e s_e w_e X_e \right) \right\|$$

Constructing Eulerian Sparsifiers?

Pick random signing $s_e \sim \pm 1$ and linearly project s to ensure Eulerianness

Challenge 2:

$$\max_e P(s)_e \sim \sqrt{\log m}$$

Scale down $s_e = \pm \frac{1}{\sqrt{\log m}}$

Constructing Eulerian Sparsifiers?

Pick random signing $s_e = \pm \frac{1}{\sqrt{\log m}}$ and linearly project s to ensure Eulerianness

Challenge 3: The sparsity is not reducing!

Key Lemma: Repeating this $\sim \log n$ times leads to $\Omega(m)$ coordinates close to 0

Potential argument: show that $\sum_e \log w'_e/w_e$ is decreasing in expectation

Bounding the variance

Want to prove

$$|x^\top \vec{L}_H y - x^\top \vec{L}_G y| \leq \delta \cdot \sqrt{x^\top U_G x \cdot y^\top U_G y}$$

Suffices

$$\left\| U_G^{-1/2} (\vec{L}_H - \vec{L}_G) U_G^{-1/2} \right\| \leq \delta$$

Bounding the variance

Want to prove

$$|x^\top \vec{L}_H y - x^\top \vec{L}_G y| \leq \delta \cdot \sqrt{x^\top U_G x \cdot y^\top U_G y}$$

Suffices

$$\left\| \text{Var} \left(\sum_e P(s)_e w_e U_G^{-1/2} \vec{L}_e U_G^{-1/2} \right) \right\| \leq \frac{\delta^2}{\log n}$$

Bounding the variance

Want to prove

$$|x^\top \vec{L}_H y - x^\top \vec{L}_G y| \leq \delta \cdot \sqrt{x^\top U_G x \cdot y^\top U_G y}$$

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Bounding the variance

Want to prove

$$|x^\top \vec{L}_H y - x^\top \vec{L}_G y| \leq \delta \cdot \sqrt{x^\top U_G x \cdot y^\top U_G y}$$

Suffices

$$\left\| \text{Var} \left(\sum_{e:u \rightarrow v} s_e w_e U_G^{-1/2} (\chi_u - \chi_v) \chi_u^\top U_G^{-1/2} \right) \right\| \leq \frac{\delta^2}{\log n}$$

Bounding the variance

Want to prove

$$|x^\top \vec{L}_H y - x^\top \vec{L}_G y| \leq \delta \cdot \sqrt{x^\top U_G x \cdot y^\top U_G y}$$

Suffices

$$\left\| \sum_{e:u \rightarrow v} w_e^2 U_G^{-1/2} (\chi_u - \chi_v) \chi_u^\top U_G^{-1} \chi_u (\chi_u - \chi_v)^\top U_G^{-1/2} \right\| \leq \frac{\delta^2}{\log n}$$

Bounding the variance

Want to prove

$$|x^\top \vec{L}_H y - x^\top \vec{L}_G y| \leq \delta \cdot \sqrt{x^\top U_G x \cdot y^\top U_G y}$$

Suffices

$$\max_{e:u \rightarrow v} w_e \chi_u^\top U_G^{-1} \chi_u \cdot \left\| \sum_{e:u \rightarrow v} w_e U_G^{-1/2} (\chi_u - \chi_v) (\chi_u - \chi_v)^\top U_G^{-1/2} \right\| \leq \frac{\delta^2}{\log n}$$

Bounding the variance

Want to prove

$$|x^\top \vec{L}_H y - x^\top \vec{L}_G y| \leq \delta \cdot \sqrt{x^\top U_G x \cdot y^\top U_G y}$$

Suffices

$$\max_{e:u \rightarrow v} w_e \chi_u^\top U_G^{-1} \chi_u \leq \frac{\delta^2}{\log n}$$

Undirected

$$\max_{e:uv} w_e \cdot (\chi_u - \chi_v)^\top U_G^{-1} (\chi_u - \chi_v)$$

$$\max_e w_e \cdot r_e$$

Use small leverage score edges

Directed

$$\max_{e:u \rightarrow v} w_e \chi_u^\top U_G^{-1} \chi_u$$

Bounded by $\left(\max_e w_e \right) \left(\max_{u,v} r_{u,v} \right)$

Decompose the graph into small effective resistance diameter pieces

Effective Resistance Decomposition

Goal: Given a graph, find edge-disjoint pieces such that in every piece,

$$\left(\max_e w_e \right) \left(\max_{u,v} r_G(u, v) \right)$$

is small, and at least half the edges are contained in the pieces.

[Jambulapati-S-Sidford-Tian-Zhao'24]

In $m \log n$ time, can find such a decomposition with quality $\frac{n}{m} \log n$

Efficient Eulerian Sparsification

[Jambulapati-S-Sidford-Tian-Zhao'24]

In $m \log^3 n$ time, can construct Eulerian sparsifiers with $n \log^2 n$ edges

[Peng-Song'22, Jambulapati-S-Sidford-Tian-Zhao'24]

Can solve directed Laplacian systems in $m \log^3 n + n \log^6 n$ time

Open Questions

Are there Eulerian sparsifiers with $n \log n$ edges?

With $O(n)$ edges? Known for undirected [[Batson-Spielman-Srivastava'08](#)]

Is there an effective resistance decomposition with quality $\frac{n}{m}$?

Yes for undirected [[Alev-Anari-Lau-Gharan'18](#)]

Thanks!