

CSC2/455 Software Analysis and Improvement

Introduction to Hoare Logic

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URCS

Outline

Logics

A Logic for Proofs of Programs

Program Verification using Hoare Logic

Postscript

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- OED Definition: Reasoning conducted or assessed according to strict principles of validity.
- Particularly relevant to this lecture:
 - A particular system or codification of the principles of proof and inference.

Propositional Logic

- Recall, propositions (identified by symbols)
 - The connectives $\vee, \wedge, \implies, \iff$ and the operation \neg
- Tautologies
 - A formula that is always true
- Contradiction
 - A formula that is always false
- Equivalence: two formulae A and B are equivalent if $A \iff B$ is a tautology
- “Proof” technique in propositional logic
 - Enumerate all possible values of variables and check if the final result is always true

Equivalences

- $p \implies q$ is equivalent to $\neg q \implies \neg p$
 - contrapositive (theorem)
- $p \implies q$ is not necessarily equivalent to $q \implies p$
 - converse

Valid Arguments

$$\begin{array}{c} P_1 \\ P_2 \\ \dots \\ P_n \\ \hline P_{n+1} \end{array}$$

- An argument is valid if and only if $P_1 \wedge P_2 \wedge \dots \wedge P_n \implies P_{n+1}$ is a tautology
 - this means that $P_1 \wedge P_2 \wedge \dots \wedge P_n \wedge P_{n+1}$ is true

Rules of Inference: Modus Ponens

$$\begin{array}{c} p \\ p \implies q \\ \hline q \end{array}$$

- $(p \wedge (p \implies q)) \implies q$ is a tautology
- Example:
 - p is “it is raining”
 - $p \implies q$ is “if it is raining, roads are wet”
 - q , so “roads are wet”

Rules of Inference: Modus Tollens

$$p \implies q$$

$$\neg q$$

$$\neg p$$

- $((p \implies q) \wedge (\neg q)) \implies \neg p$ is a tautology
- Example:
 - $p \implies q$ is “if a is even, $a + 1$ is odd”
 - $\neg q$ is “ $a + 1$ is not odd”
 - $\neg p$, so “ a is not even”

Invalid Rule of Inference: Affirming the Consequent

$$\begin{array}{c} p \implies q \\ q \\ \hline p \end{array}$$

- $((p \implies q) \wedge q) \implies p$ is *not* a tautology

Proof System for Propositional Logic

- System L
- Lines of proof in this system must be
 - an axiom of L (an axiom of L is a tautology)
 - an application of Modus Ponens
 - a hypothesis (a hypothesis G_n is assumed to be true)
 - a lemma (a previously proven theorem)
- The last line of a proof is a theorem
 - $G_1, G_2, \dots, G_n \vdash_L A$
- This proof system is both:
 - Sound: Only tautologies can be proved
 - Complete: All tautologies can be proved

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Floyd-Hoare Logic

Developed by Robert Floyd and Tony Hoare in the 1960s.

$$\{P\}C\{Q\}$$

- P is a precondition
- C is a statement, function or program
- Q is a postcondition
- Both P and Q are logical statements, e.g., what you would put in an assert

Read as: If P holds, and C executes (and terminates), then Q holds. P and Q are assertions, usually over program state, and usually we need to prove that Q holds.

Recall: Partial and Total Correctness

- If C does not terminate, Q may or may not be true
 - This is the notion of *partial correctness*
- If C can be shown (formally) to terminate, then we achieve a proof of *total correctness*

Total correctness = Termination + Partial Correctness

Some examples of assertions

- $\{X = 1\} Y := X \{Y = 1\}$
- $\{X = 1\} Y := X \{Y = 2\}$
- $\{\text{true}\} C \{Q\}$
- $\{P\} C \{\text{true}\}$
- $\{P\} C \{\text{false}\}$

Note: not all of the above are valid, they are just assertions to be checked.

Formal Proof

- (informally) Proofs at the level of rigour that even a computer could understand!
- Usually, each step in the proof is *explicitly* annotated as to how it was obtained from the previous steps
 - Makes it easy to check (esp. for computers)
 - Either the use of an *axiom* or a *rule of inference*
- Painful to construct by hand
 - Interactive proof assistants like Coq, Isabelle, and Lean usually make it more fun
 - (if you've disliked writing proofs, try them!)

The assignment axiom of Hoare Logic

- The *assignment axiom* states that
 - $\vdash \{P[E/V]\} V := E \{P\}$
- $P[E/V]$ is read as P with all instances of V replaced by E
 - P with E for V
 - $\{X = 1\}[Y/X]$ leads to $\{Y = 1\}$
- Usage example: if $X = 6$, prove $Y > 15$ after $Y := X * 3$
 - Postcondition P to prove: $\{Y > 15\}$
 - Use assignment axiom: $\{X \times 3 > 15\} Y := X * 3 \{Y > 15\}$
 - Given that $X = 6$, so $X \times 3 = 6 \times 3 = 18$
 - $X * 3 = 18 \implies X * 3 > 15$

Two incorrect assignment axiom forms

- $\{P\} V := E \{P[E/V]\}$
- $\{P\} V := E \{P[V/E]\}$

Precondition strengthening

If $\vdash \{P'\} C \{Q\}$ and $P \implies P'$, then we can write $\vdash \{P\} C \{Q\}$

- $\{X + 1 = n + 1\} X := X + 1 \{X = n + 1\}$ (assignment axiom)
- $\vdash X = n \implies X + 1 = n + 1$ (from arithmetic)
- $\{X = n\} X := X + 1 \{X = n + 1\}$ (precondition strengthening)
- P' is known as the *weakest precondition*, and is unique.

Postcondition weakening

If $\vdash \{P\} C \{Q'\}$, and $Q' \implies Q$, then we can write $\vdash \{P\} C \{Q\}$

- $\{R = X \wedge 0 = 0\} Q := 0 \{R = X \wedge Q = 0\}$ (assignment axiom)
- $R = X \wedge Q = 0 \implies R = X + (Y \times Q)$
- $\{R = X\} Q := 0 \{R = X + (Y \times Q)\}$ (postcondition weakening)

Conjunctions and Disjunctions

- If $\vdash \{P_1\} \text{C} \{Q_1\}$ and $\vdash \{P_2\} \text{C} \{Q_2\}$, then $\vdash \{P_1 \wedge P_2\} \text{C} \{Q_1 \wedge Q_2\}$
- If $\vdash \{P_1\} \text{C} \{Q_1\}$ and $\vdash \{P_2\} \text{C} \{Q_2\}$, then $\vdash \{P_1 \vee P_2\} \text{C} \{Q_1 \vee Q_2\}$

Sequencing Rule

- If $\vdash \{P\} C1 \{Q\}$ and $\vdash \{Q\} C2 \{R\}$, then $\vdash \{P\} C1; C2 \{R\}$
- You can combine the sequencing rule and the *rules of consequence* (i.e. precondition strengthening and postcondition weakening) to extend this to multiple statements.

The Conditional Rule

- If $\vdash \{P \wedge S\} C1 \{Q\}$ and $\vdash \{P \wedge \neg S\} C2 \{Q\}$, then
 - $\vdash \{P\} \text{ IF } S \text{ THEN } C1 \text{ ELSE } C2 \{Q\}$

The While Rule

- If $\{P \wedge S\} C \{P\}$ then
 - $\vdash \{P\} \text{ WHILE } S \text{ DO } C \text{ ENDDO } \{P \wedge \neg S\}$
- Here, P is the *inductive loop invariant*, recall:
 - It is true on entry into and exit out of the loop
 - It is true after every iteration of the loop

More rules

- FOR-rule
- Handling arrays
 - variant of assignment, due to McCarthy

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Example 1

$$X = x \wedge Y = y$$

R := X;

X := Y;

Y := R;

$$X = y \wedge Y = x$$

Generating Verification Conditions

- A verification condition is a mechanically generated proof goal from the program and program specifications.
- For example, suppose $\{P\} V := E \{Q\}$ exists in the program
 - P is programmer-supplied precondition (or annotation)
 - Q is programmer-supplied postcondition
- The verification condition for this statement is
$$P \implies Q[E/V]$$

Why the VC for assignment works

- From Hoare Logic, we have:
 - $\vdash \{Q[E/V]\} v := E \{Q\}$
- If we prove $P \implies Q[E/V]$, then by precondition strengthening, we have:
 - $\vdash \{P\} v := E \{Q\}$
- Which is what we had to prove.

What if we can't prove $P \implies Q[E/V]$? Does that mean $\{P\} C \{Q\}$ does not hold?

Sufficiency and Incompleteness

- VCs are *sufficient*, but not necessary
 - There may be other ways to prove $\{P\}C\{Q\}$
- Mechanical provers cannot prove everything
 - Gödel's Incompleteness Theorem

Verification conditions for our example

$$\begin{array}{l} \{X = x \wedge Y = y\} \quad R := X; \\ \quad X := Y; \\ \quad Y := R; \quad \{X = y \wedge Y = x\} \end{array}$$

- The verification conditions for a sequence ending in an assignment $\{P\} C1; V := E \{Q\}$ are those generated by:
 - $\{P\} C1 \{Q[E/V]\}$

Verification conditions for our example: 2

$$\begin{array}{l} \{X = x \wedge Y = y\} \quad R := X; \\ \quad X := Y; \quad \{X = y \wedge R = x\} \end{array}$$

- Because $\{X = y \wedge Y = x\}[R/Y]$, following from VC for sequences ending in an assignment.

Verification conditions for our example: 3

$$\{X = x \wedge Y = y\} \quad R := X; \quad \{Y = y \wedge R = x\}$$

- $P = \{X = x \wedge Y = y\}$
- $Q = \{Y = y \wedge R = x\}$
- Using VC for assignment:
 - $Q[E/V] = \{Y = y \wedge R = x\}[X/R] = \{Y = y \wedge X = x\}$
- Here, $P \implies Q[E/V]$ trivially (identical)

Example 2

$$k \geq 0$$

```
x := k;  
c := 0;  
  
while(x > 0) {  
    x := x - 1;  
    c := c + 1;  
}
```

$$x = 0 \wedge c = k$$

Verification conditions for While and Sequences

- The verification conditions for a While statement $\{P\} \text{ WHILE } S \text{ DO } C \{Q\}$ are
 - $P \implies R$ (where R is the loop invariant)
 - $R \wedge \neg S \implies Q$
 - recursively, all VCs from $\{R \wedge S\} C \{R\}$
- The verification conditions for a sequence not ending in an assignment $\{P\} C_1; C_2; C_{(n-1)}; C_n \{Q\}$, assuming $\{R\} C_{(n)} \{Q\}$ are those generated by:
 - $\{R\} C_n \{Q\}$
 - $\{P\} C_1; C_2; C_{(n-1)} \{R\}$

Verification Conditions for While loop and body

```
while(x > 0) {  
    x := x - 1;  
    c := c + 1;  
}  
/* Q: x = 0 /\ c = k */
```

- loop invariant: $x + c = k$
- (VC1) $x + c = k \wedge \neg(x > 0) \implies x = 0 \wedge c = k$
 - (from $R \wedge \neg S \implies Q$)
- (VC2) $P \implies x + c = k$ (from $P \implies R$)
- (VC3) $x + c = k \wedge x > 0 \implies x - 1 + c + 1 = k$ (VC from assignment)
 - Recursively from body:
 - $\{x + c = k \wedge x > 0\} x := x - 1; c := c + 1 \{x + c = k\}$
 - $\{x + c = k \wedge x > 0\} x := x - 1 \{x + c + 1 = k\}$ (from sequence ending with assignment)

Verification Conditions for Initialization

```
/* k >= 0 */  
x := k;  
c := 0;  
/* P */
```

- Let's assume $P = R$, so P is $x + c = k$
- (VC0) $k \geq 0 \implies k = k$
 - $\{k \geq 0\} x := k; c := 0 \{x + c = k\}$
 - $\{k \geq 0\} x := k; \{x + 0 = k\}$ (from sequence ending with assignment)
 - $Q[E/V]$ is $k + 0 = k$

Verification Conditions

- (VC0) $k \geq 0 \implies k = k$
- (VC1) $x + c = k \wedge \neg(x > 0) \implies x = 0 \wedge c = k$
- (VC2) $x + c = k \implies x + c = k$
- (VC3) $x + c = k \wedge x > 0 \implies x + c = k$
- We need to show that $VC_0 \wedge VC_1 \wedge VC_2 \wedge VC_3$ is valid.

Negate and UNSAT

- Implication $P \implies Q$ can be read as, assume P is true, show that Q is valid.
 - $P \vdash Q$
- For proving with SAT solvers, if $P \implies Q$, check whether $P \wedge \neg Q$ is unsatisfiable
 - $P \implies Q$ can also be written as $\neg P \vee Q$

SMT to the rescue

```
from z3 import *

s = Solver()
x, k, c = Ints('x k c')

vc0 = Implies(k >= 0, k == k)
vc1 = Implies(And(x + c == k, Not(x > 0)), And(x == 0, c == k))
vc2 = Implies(x + c == k, x + c == k)
vc3 = Implies(And(x + c == k, x > 0), x + c == k)

s.add(x >= 0)
s.add(Not(And(And(And(vc0, vc1), vc2), vc3)))

if s.check() == sat:
    print("SAT", s.model())
else:
    print("UNSAT")
```


Program Verification Procedure

- Generate specifications (aka annotations or assert statements)
- Generate verification conditions
 - Usually mechanical, e.g. Dafny or CBMC
- Prove verification conditions
 - By hand or
 - Automated Theorem Prover

More stuff

- Generating VCs for other statements in language
- Soundness?
- Completeness?
- Decidability?
- Pointers: Separation logic

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Sources, further reading and links

- Background Reading on Hoare Logic, by Mike Gordon
 - The reference for this lecture
- Textbooks
 - Software Foundations: Vol 1: Logical Foundations,
 - Software Foundations: Vol 2: Programming Language Foundations
 - Concrete Semantics