CSC2/455 Software Analysis and Improvement
Array/Loop Dependence Analysis

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URCS

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Outline

Characterizing loop dependences

Identifying Loop Dependences

Distance and Direction Vectors

Postscript
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Postscript
Definition: Two *dynamic* statements have a dependence if:
- Both access same location (memory or register)
- And one of the accesses is a write

*Dynamic* required, since we’re talking about loops (examples later)
Dependence types

\( S_1 \) occurs “earlier” than \( S_2 \) (in the dynamic trace)

- True dependence
  - \( S_1 \delta S_2 \)
  - \( S_1 \) writes, \( S_2 \) reads

- Anti-dependence
  - \( S_1 \delta^{-1} S_2 \)
  - \( S_1 \) reads, \( S_2 \) writes

- Output dependence
  - \( S_1 \delta^o S_2 \)
  - Both \( S_1 \) and \( S_2 \) write
Loop-independent dependence

What are the dependences in the loop body below?

Can you change the order of the statements in the loop body?

```
DO I = 0, 9
   A(I) = A(I) + B
   C(I) = A(I) + D
ENDDO
```

Can you change the (execution) order of loop iterations?

Note: FORTRAN uses parentheses in array references: e.g., A(I). FORTRAN arrays usually start at 1, but for this lecture, we will assume they start at 0.
Loop-independent dependences visualized

NOTE: Only dependences from first four iterations visualized.
Loop-carried dependences

What are the dependences in the loop body below?

Can you change the order of the statements in the loop body?

```
DO I = 0, 9
   A(I + 1) = A(I) + B
   C(I) = A(I) + D
ENDDO
```

Can you change the (execution) order of loop iterations?
Loop-carried dependences visualized

A(0 + 1) = A(0) + B
A(0 + 1) = A(0) + B
A(1 + 1) = A(1) + B
A(1 + 1) = A(1) + B
A(2 + 1) = A(2) + B
A(2 + 1) = A(2) + B
C(1) = A(1) + D
C(1) = A(1) + D
C(2) = A(2) + D
C(2) = A(2) + D
C(0) = A(0) + D

NOTE: Only dependences from first three iterations visualized.
DO I = 0, 9
  DO J = 0, 1
    A(I + 1, J) = A(I, J) + 1
  ENDDO
ENDDO

- Can you change the order of inner loop?
- Can you change the order of the outer loop?
Dependences Visualized

\[ A(0 + 1, 0) = A(0, 0) + 1 \]
\[ A(1 + 1, 0) = A(1, 0) + 1 \]
\[ A(2 + 1, 0) = A(2, 0) + 1 \]
\[ A(0 + 1, 1) = A(0, 1) + 1 \]
\[ A(1 + 1, 1) = A(1, 1) + 1 \]
\[ A(2 + 1, 1) = A(2, 1) + 1 \]

NOTE: Only dependences from first three iterations visualized.
Loop Dependences

- Loop-independent dependence
  - In same iteration, independent of loops
- Loop-carried dependence
  - Across different iterations of atleast one loop
- Dependence Level of a Loop-carried Dependence
  - The nesting level $k$ of loop that carries the dependence
  - $S_1\delta_k S_2$
Iteration Spaces

DO I = 1, 2
    DO J = 1, 2
        S
    ENDDO
ENDDO

▶ S has four instances \((I, J)\): \((1, 1)\), \((1, 2)\), \((2, 1)\), \((2, 2)\)
▶ Each of these values represents an iteration vector
    ▶ Particular values of loop indices
    ▶ Ordered from outermost loop to innermost loop
DO J = 1, 10
    DO I = 1, 10
        A(I+1, J) = A(I, J) + X
    ENDDO
ENDDO

Assuming A starts from 1. FORTRAN allows you to change the "origin" of arrays.
Iteration Space Figure

[Diagram of iteration space figure with numbers from 1,1 to 10,10]

1,1  2,1  3,1  4,1  5,1  6,1  7,1  8,1  9,1  10,1
For two vectors $i = (i_1, i_2, ..., i_n)$ and $j = (j_1, j_2, ..., j_n)$, each containing $n$ elements, $i < j$ if there exists $m \in [0, n)$, such that:

- $i_x = j_x$ for $x < m$
- $i_m < j_m$
Iteration Vector Ordering (Code)

For two vectors $i$ and $j$, each containing $n$ elements, $i < j$ is defined as:

```python
def lessthan(i, j, n):
    if n == 1:
        return i[0] < j[0]

    # test prefix for elementwise-equality
    if i[0:n-1] == j[0:n-1]:
        return i[n-1] < j[n-1]
    else:
        return lessthan(i, j, n-1)
```

Can similarly define other order relations.
Loop dependence

Dependence from Statement S1 (source) to statement S2 (sink) if:

- There exist iteration vectors $i$ and $j$ such that $i < j$ or $i = j$
- There is a path from S1 to S2 in the loop
- S1 accesses memory location M in iteration $i$
- S2 accesses memory location M in iteration $j$
- and one of the accesses is a write
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Generalizing Loop Indices

DO I_1 = ...  
  DO I_2 = ...  
  ...  
  DO I_N = ...  
    A(f1, f2, f3, ..., fM) = ...  
    ... = A(g1, g2, g3, ..., gM)  
  ENDDO  
ENDDO  
ENDDO

where A is \( M \)-dimensional array, and \( f_X \) and \( g_X \) are index functions of the form

- \( f_X(I_1, I_2, ..., I_N) \)
- \( g_X(I_1, I_2, ..., I_N) \)
- \( 1 \leq X \leq M \)
Dependence using Iteration Vectors

Let \( \alpha \) and \( \beta \) be iteration vectors:

- \[ \alpha = (i_1, i_2, i_3, ..., i_N) \]
- \[ \beta = (i'_1, i'_2, i'_3, ..., i'_N) \]

Then a dependence exists if:

- \( \alpha < \beta \) (vectors)
- \( f_X(\alpha) = g_X(\beta) \), for \( 1 \leq X \leq M \)
Example

```plaintext
DO J = 0, 9
    DO I = 0, 9
        A(I+1, J) = A(I, J) + X
    ENDDO
ENDDO
```

- $f_1(J, I) = I + 1$, $f_2(J, I) = J$
- $g_1(J, I) = I$, $g_2(J, I) = J$
- For $\alpha = (0, 0)$ (i.e. $J = 0$, $I = 0$) and $\beta = (0, 1)$ (i.e. $J = 0$, $I = 1$):
  - $f_1(\alpha) = g_1(\beta)$, i.e. $1 = 1$
  - $f_2(\alpha) = g_2(\beta)$, i.e. $0 = 0$
  - Many other values of $\alpha$ and $\beta$ also satisfy these equations.
Dependence Testing

Do iteration vectors $\alpha$ and $\beta$ exist such that:

- (vectors) $\alpha < \beta$
- $f_X(\alpha) = g_X(\beta)$, for $1 \leq X \leq M$

How can we find $\alpha$ and $\beta$ if they exist?
Restrictions on Index functions

- \( fX \) and \( gX \) must be decidable (i.e. computable)
- \( fX \) and \( gX \) must be "analyzable"
  - to avoid brute force search
Let $f_X$ and $g_X$ must be affine functions of loop indices:

- i.e. for $f_X(i_1, i_2, i_3, ..., i_n)$
- $f_X = a_1 i_1 + a_2 i_2 + ... + a_n i_n + e$
- $e$ is optional loop invariant calculation (i.e. constant for the loop)
Given that $f_X$ and $g_X$ are affine functions of loop indices
Do iteration vectors $\alpha$ and $\beta$ exist such that:

- (vectors) $\alpha < \beta$
- $f_X(\alpha) = g_X(\beta)$, for $1 \leq X \leq M$

How can we find $\alpha$ and $\beta$ if they exist?
What is this problem better known as? Hint: an affine function is a linear function plus constant.
Dependence Testing

- Integer Linear Programming is NP-complete
- Lots of heuristics invented
  - Profitable to know if no solution exists since it implies no dependence!
  - See Chapter 3 of AK
  - Or Chapter 11 of the Dragon Book
  - We will cover this in a later class
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Representing Dependences

Do we need to track all the iterations that have a dependence explicitly (e.g. in a list)?
Distance Vectors

\[ d(i, j)_k = j_k - i_k \]

- Where \( i, j, d(i, j) \) are \( n \)-element vectors
- \( i_k \) indicates \( k \)-th element of \( i \)

Example distance vector: \((0, 1)\)
Direction Vectors

\[ D(i, j)_k = \]
- "<", if \( d(i, j)_k > 0 \)
- "=" , if \( d(i, j)_k = 0 \)
- ">" , if \( d(i, j)_k < 0 \)

Example direction vector for \((0, 1)\): \((=, <)\)
Information we need to track

For every pair of memory references:

- Iteration Vectors $i$ and $j$ which have a dependence, or
- Unique Distance Vectors $d(i, j)$, or
- Unique Direction Vectors $D(i, j)$
Which of these indicates a loop-independent dependence?

- $(=, =)$
- $(=, <)$

Of the loop-carried dependence in example above, what level is the loop-carried dependence?
Theorems

WARNING: Informal language

▶ Direction Vector Transform (Theorem 2.3 in AK)
  ▶ If a transformation reorders loop iterations, and preserves the leftmost non-“=” component as “<”, all dependences are preserved.

▶ Theorem 2.4 in AK
  ▶ If a level-k dependence exists, and a transformation reorders loop iterations while not reordering the level-k loop
  ▶ And does not move loops inside k outside the loop and vice versa
  ▶ It preserves all level-k dependences.

▶ Iteration Reordering (Theorem 2.6 in AK)
  ▶ Iterations of a level $k$ loop can be reordered if there is no level $k$ dependence.
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References

- Much of this lecture is based on AK, Chapter 2.
- Chapter 11 of the Dragon Book also presents this information, but differently.