Outline

Review

Proofs

Constant Propagation

Postscript
Part I of Foundations

- Methods to solve dataflow analysis equations
  - IDEAL
  - Meet over paths (MOP)
  - Maximum Fixed Point (MFP)
  - $\text{IDEAL} \subseteq \text{MOP} \subseteq \text{MFP}$

- (Semi)lattice-based framework
  - $(D, V, \wedge, F)$, dataflow analysis
  - $(V, \wedge)$, meet semilattice
  - $(V, \leq)$, partial order, where $x \leq y$ iff $x \wedge y = x$
  - Monotone framework

- Greatest Lower Bound
  - $z \leq x$ and $z \leq y$, where $z = x \wedge y$
A given \((D, V, \land, F)\) is monotone if for all \(x, y \in V\), and \(f \in F\):

- \(x \leq y \rightarrow f(x) \leq f(y)\)
- equivalently, \(x \leq y \rightarrow f(x \land y) \leq f(x) \land f(y)\)
- The proof of equivalence is in the textbook.

In addition, the framework is *distributive* if:

- \(f(x \land y) = f(x) \land f(y)\)

Note that these properties do not necessarily arise automatically, \(F\) must be designed to have these properties.

- And proofs must be written to show that \(F\) does.
- We’ll see this for a complicated example today.
General Iterative Algorithm

forwards(IN, OUT, meet, top, v_entry, f_transfer)
  OUT[entry] = v_entry

  for each basic block B except ENTRY:
    OUT[B] = top

  do {
    for each basic block B except ENTRY:
      # this calculates the meet over predecessors, \( \wedge \ p \) OUT[p]
      IN[B] = reduce(meet, [OUT[p] for p in B.predecessors])
      OUT[B] = f_transfer(IN[B])
  } while(some OUT changes value)

- Does this calculate the solution to the dataflow problem?
- Does this algorithm terminate?
- Does this algorithm calculate the maximum fixed point – i.e. the most precise solution admissible?
This class

- Proofs that answer these three questions
- Relationships between IDEAL, MOP and MFP in terms of the framework
- Examples of:
  - a non-distributive framework (from Dragon 9.4, Constant Propagation)
  - lattices containing infinite values
  - possibly some proof writing exercises (from Dragon 9.3)
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do {
    for each basic block B except ENTRY:
        # this calculates the meet over predecessors, \p OUT[p]
        IN[B] = reduce(meet, [OUT[p] for p in B.predecessors])
        OUT[B] = f_transfer(IN[B])
} while(some OUT changes value)

The iterative algorithm computes the solution to the dataflow problem.

- The iterative algorithm performs an unbounded number of iterations as long as IN and OUT change
- *When it terminates*, IN and OUT have not changed for an iteration
- The values of IN and OUT therefore satisfy the equations
  - Hence they are solutions of the dataflow problem
Proof #2

The iterative algorithm terminates (i.e. converges to a fix point).

- When we apply the $\land$ operator, we obtain the glb
  - i.e. $z = x \land y$ and $z \leq x$ and $z \leq y$
- Since the framework is monotone:
  - $f(x) \leq f(y)$ if $x \leq y$
  - i.e. OUT values are no greater than the IN values
- At each step, these values decrease or remain the same
  - When they all remain the same, we terminate
- If values decrease, recall the lattice has finite height
  - Implies a finite number of steps before we reach $\bot$
  - $x \land \bot = \bot$ and $f(\bot) = \bot$ (i.e once a value becomes $\bot$, it no longer changes)
  - We terminate in this case as well
The fixed point solution computed by the iterative algorithm is the \textit{maximum} fixed point.

\textbf{Proof} By induction, for forward analyses

\textit{(BASIS)} After the first iteration, values of IN\textit{[B]} and OUT\textit{[B]} are \(\leq\) their initial values.

- At initialization, OUT\textit{[B]} is \(\top\) for all blocks \(B\) except ENTRY
- After the first iteration, in a monotone framework, all values will be \(\leq\) those at initialization by definitions of the \(\land\) and transfer functions
Proof #3: Inductive step

Assume that:

- $\text{IN}[B]^k \leq \text{IN}[B]^{k-1}$
- $\text{OUT}[B]^k \leq \text{OUT}[B]^{k-1}$

Show that:

- $\text{IN}[B]^{k+1} \leq \text{IN}[B]^k$
- $\text{OUT}[B]^{k+1} \leq \text{OUT}[B]^k$
Proof #3: Continued

- To obtain $\text{IN}[B]$ we must apply $\wedge$ to all $\text{OUT}[P]$
  - $P$ is a predecessor of $B$
  - This implies $\text{IN}[B] \leq \text{OUT}[P]$ ($\wedge$ yields glb)
  - From our inductive hypothesis, $\text{OUT}[P]^k \leq \text{OUT}[P]^{k-1}$
  - Applying $\wedge$ on both sides over all $P$, $\text{IN}[B]^{k+1} \leq \text{IN}[B]^k$

- Now, $\text{OUT}[B] = f(\text{IN}[B])$
  - In the monotone framework, $f(x) \leq f(y)$ when $x \leq y$
  - We have shown $\text{IN}[B]^{k+1} \leq \text{IN}[B]^k$
  - Therefore, after applying $f$ to both sides, by monotonicity, we have $\text{OUT}[B]^{k+1} \leq \text{OUT}[B]^k$
Properties of the IDEAL solution

- Any solution greater than IDEAL is incorrect (or unsafe)
- Any solution less than or equal to IDEAL is conservative, or safe.

To see why, consider IDEAL solution $x = p_1 \land p_2 \land ... \land p_n$:

- How can we obtain a value $z = p_1 \land ... \text{ greater than } x$?
- How can we obtain a value $y = p_1 \land ... \text{ less than } x$?

(recall the relationship between the results of the meet operator and its operands)

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1 In the English sense
Relationship between IDEAL and MOP

- MOP considers a superset of all executable paths
  - MOP solution $y = p_1 \land p_2 \land \ldots \land p_n \land p_{n+1} \ldots$
- What is the relationship between MOP ($y$) and IDEAL ($z$)?
Relationship between MOP and MFP

- \( \text{MOP}[B_4] = ((f_{B_3} \circ f_{B_1}) \land (f_{B_3} \circ f_{B_2}))(v_{entry}) \)
  - i.e., compose transfer functions over a path and then apply meet (e.g. \( f_{B_3}(f_{B_1}(v_{entry})) \))

- \( \text{IN}[B_4] = f_{B_3}(f_{B_1}(v_{entry}) \land f_{B_2}(v_{entry})) \)
  - i.e. apply meet at join nodes
In a distributive framework, MOP = MFP

- \( \text{MOP}[B_4] = ((f_{B_3} \circ f_{B_1}) \land (f_{B_3} \circ f_{B_2}))(v_{\text{entry}}) \)
- \( \text{IN}[B_4] = f_{B_3}(f_{B_1}(v_{\text{entry}}) \land f_{B_2}(v_{\text{entry}})) \)

If \( f(x \land y) = f(x) \land f(y) \) (i.e. distributive):

- \( \text{IN}[B_4] = f_{B_3}(f_{B_1}(v_{\text{entry}})) \land f_{B_3}(f_{B_2}(v_{\text{entry}})) \)

- If the framework is distributive, then MOP solution = MFP solution
  - Otherwise by monotonicity MFP \( \leq \) MOP

- In either case,
  - MFP \( \leq \) MOP \( \leq \) IDEAL
  - So all methods produce “safe” solutions
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Analyses so far

- Live variable analysis
- Available Expressions
- Reaching Definitions
- These are all distributive (implies monotonicity)
- Their lattices contain a finite number of values
- Their lattices have finite height
Constant Propagation

- Does this variable have a constant value at this point in the program?
  - Used to perform constant folding (i.e. evaluate constant expressions at compile time)
- Data flow analysis framework
  - Direction?
  - Values?
  - Meet operator?
  - Transfer function?
Constant Propagation

- Direction: Forward
- Values:
  - UNDEF: variable is undefined so far
  - $c$: variable is constant value $c$
  - NAC: variable is not a constant
- Meet operators and transfer functions are slightly more complicated.
Meet for Constant Propagation

- UNDEF $\land v = ?$
- NAC $\land v = ?$
- $c \land c = ?$
- $c_1 \land c_2 = ? (c_1 \neq c_2)$
Meet for Constant Propagation

- \( \text{UNDEF} \land v = v \)
  - \( \text{UNDEF} \) is \( \top \)
- \( \text{NAC} \land v = \text{NAC} \)
  - \( \text{NAC} \) is \( \perp \)
- \( c \land c = c \)
- \( c_1 \land c_2 = \text{NAC} \)

What does the lattice for constant propagation look like?
The lattice for constant propagation
The Transfer Function

- \( \text{OUT}[s] = f(\text{IN}[s]) \) for a statement \( s \)
  - Slightly easier to understand if we use statements instead of basic blocks

- Observe that non-assignment statements do not change values
  - \( f \) is simply the identify function \( f(x) = x \) for such statements

- What about assignment statements?
  - \( x = c \), where \( x \) is a variable, and \( c \) is a constant
  - \( x = y + z \), where + is any binary operator
  - \( x = *y \) or \( x = f(\ldots) \), where \( f \) is a function call
Note that IN (and OUT) are maps (i.e. dictionaries)
- From variables to their current dataflow values (UNDEF, c, or NAC)
- Let’s call this map $m$, so that $m(x)$ returns the dataflow value for variable $x$

- $x = c$, changes $m(x) \leftarrow c$
- $x = y + z$, where $+$ is any binary operator (not just addition)
  - $m(x) \leftarrow m(y) + m(z)$ if $m(y)$ and $m(z)$ are constants
  - $m(x) \leftarrow \text{NAC}$ if either $m(y)$ or $m(z)$ is NAC
  - $m(x) \leftarrow \text{UNDEF}$ otherwise
- $x = \ast y$ or $x = f(...)$, $m(x) \leftarrow \text{NAC}$ (conservatively)
- Note that $m(v) \leftarrow m(v)$ for all $v \neq x$
  - I.e. the other values of the map remain unchanged

Note that I use slightly different notation than the textbook, which uses $m'$ on the LHS
Is this monotonic?

Is \( \text{OUT}[s] \leq \text{IN}[s] \) for every \( s \)?

- For the two cases below, it is “surely ... monotone”:
  - \( m(x) \leftarrow c \)
  - \( m(x) \leftarrow \text{NAC} \)
- What about \( x = y + z \)?
  - Need to show that \( m(x) \) does not get greater as \( m(y) \) (and/or) \( m(z) \) get smaller
  - Show by case analysis and symmetry
$x = y + z$ as $m(z)$ gets smaller

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<th>$m(z)$</th>
<th>output $m(x)$</th>
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<td>UNDEF</td>
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<td>$c_1$</td>
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<td>NAC</td>
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<td>$c_2$</td>
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<tr>
<td></td>
<td>NAC</td>
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</tbody>
</table>
x = y + z as m(z) gets smaller (answers)

<table>
<thead>
<tr>
<th>m(y)</th>
<th>m(z)</th>
<th>output m(x)</th>
</tr>
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<tbody>
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<td></td>
<td>NAC</td>
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</tr>
</tbody>
</table>

UNDEF

NAC

c_1

NAC

c_2

NAC
Is it distributive?

ENTRY

B1:
x = 2
y = 3

B2:
x = 3
y = 2

EXIT

B3:
z = x + y
• Path 1 \((x = 2; y = 3; z = x + y)\)
  • \(m(z) = 5\), so \(z\) is a constant
• Path 2 \((x = 3; y = 2; z = x + y)\)
  • \(m(z) = 5\), so \(z\) is a constant
• Meet over Path 1 and Path 2
  • \(m(z) = 5 \land 5\), so \(z\) is a constant
MFP solution

- At end of block $B_1$
  - $m(x) = 2$ and $m(y) = 3$
- At end of block $B_2$
  - $m(x) = 3$ and $m(y) = 2$
- Meet before block $B_3$
  - $m(x) = 2 \land 3$ (i.e. case $c_1 \land c_2$)
  - $m(y) = 3 \land 2$
- Conclusion?
For constant propagation, in most non-trivial programs
  \[ \text{MFP} < \text{MOP} \]
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References

- Chapter 9 of the Dragon book
  - Section 9.3, 9.4