Outline

Introduction

A Tiny Language and Its Semantics

To be continued ...
Introduction

A Tiny Language and Its Semantics

To be continued ...
• We learnt about program analysis tools beyond iterative dataflow analysis

• Abstract Interpretation
  • Maps concrete states of programs to abstract states
  • Abstract states belong to an abstract domain: signs, intervals, convex polyhedra, ...
  • Define transfer functions to convert pre-condition (input) states to post-condition (output) states
  • Union for alternate paths
  • Widen for loops

• This lecture:
  • Concrete Semantics for a small language
This lecture defines a number of formal concepts and is notation-heavy.

I also provide an equivalent formal notation in (Python) code to hopefully make it easier.
Introduction

A Tiny Language and Its Semantics

To be continued ...
A Tiny Language: Grammar

\[ n \in \mathbb{V} \]
\[ x \in \mathbb{X} \]
\[ \circ ::= + | - | \ast | ... \]
\[ \otimes ::= < | \leq | > | == | ... \]

- \( n \) is a set of concrete values, here we shall treat \( \mathbb{V} = \mathbb{Z} \)
  - All values are integers
- \( x \) is the name of a variable. The set \( \mathbb{X} \) contains all variable names.
- \( \circ \) represents arithmetic binary operators
- \( \otimes \) represents boolean binary operators
A Tiny Language: Expressions

\[ E ::= n \mid x \mid E \odot E \]
\[ B ::= x \oplus n \]

- An arithmetic expression \( E \) is:
  - a number, or
  - a variable name,
  - or a binary expression

- A boolean expression \( B \) is:
  - a variable,
  - a boolean operator
  - a constant \( n \)
from typing import Union
from typing_extensions import Literal

BinaryOps = Literal['+', '-', '*', '/']
ComparisonOps = Literal['<', '>', '==', '<=', '>=', '!=']

Scalar = int  # restrict Scalars to ints in this implementation

class Node(object):
    pass

class Var(Node):
    def __init__(self, name: str):
        self.name = name

    def __str__(self):
        return self.name

Expr = Union[Scalar, Var, 'BinOp']

- This is Python 3 augmented with types
  - Union stands for a union type
class BinOp(Node):
    def __init__(self, op: BinaryOps, left: Expr, right: Expr):
        self.op = op
        self.left = left
        self.right = right

... 

class BoolExpr(Node):
    def __init__(self, op: ComparisonOps, left: Var, right: Scalar):
        self.op = op
        self.left = left
        self.right = right

...

- Nothing special here, each component of the grammar is stored in the respective AST nodes
- I'm eliding implementations of __str__, indicated by '...'
Commands in the language

\[ C ::= \]

skip
| \( C; C \)
| \( x := E \)
| \( \text{input}(x) \)
| \( \text{if}(B)\{C\} \text{ else } \{C\} \)
| \( \text{while}(B)\{C\} \)

\[ P ::= C \]
class Cmd(Node):
    pass

class Skip(Cmd):
    def __init__(self):
        pass

class Seq(Cmd):
    def __init__(self, cmd0: Cmd, cmd1: Cmd):
        self.cmd0 = cmd0
        self.cmd1 = cmd1

class Assign(Cmd):
    def __init__(self, left: Var, right: Expr):
        self.left = left
        self.right = right

class Input(Cmd):
    def __init__(self, var: Var):
        self.var = var

    def __str__(self):
        return f"input({self.var})"

class IfThenElse(Cmd):
    def __init__(self, cond: BoolExpr, then_: Cmd, else_: Cmd):
        self.cond = cond
        self.then_ = then_
        self.else_ = else_

class While(Cmd):
    def __init__(self, cond: BoolExpr, body: Cmd):
        self.cond = cond
        self.body = body

class Program(Node):
    def __init__(self, cmd: Cmd):
        self.program = cmd
if(x > 7) {
    y := (x - 7)
} else {
    y := (7 - x)
}

can be represented using the AST as:

\[
\begin{align*}
x &= \text{Var('x')} \\
y &= \text{Var('y')} \\
t &= \text{Program(IfThenElse(BoolExpr('>', x, 7),}
& \quad \text{Assign(y, BinOp('-', x, 7))}, \\
& \quad \text{Assign(y, BinOp('-', 7, x))})
\end{align*}
\]
To execute programs represented as ASTs, we need the following:

- **Storage/Memory**: to track values of variables
- **Semantics**: to express what each command does, usually mathematical
  - Denotational semantics ("input/output" semantics)
  - Operational semantics
  - Axiomatic semantics
  - and many others...
M = X → V

• A store (from storage) is a map/function from variables to values
• We’ll represent it as (assuming X = {x, y}): 

\[ m = \{x \rightarrow 3, y \rightarrow 4\} \]

• Store (or memory) m maps x to 3 and y to 4.
• So, m(x) = 3, and m(y) = 4
from typing import Dict, List

# using str instead of Var, with Var.name as the key.
# This is accidental.
Memory = Dict[str, int]

x = Var('x')
y = Var('y')

m = {x.name: 3, y.name: 4}

print(m[x.name])
print(m[y.name])
• The semantics of an expression $E$ depend on the memory store $m$

• We use $\llbracket E \rrbracket(m)$ to denote its semantics

• We’ll define $\llbracket E \rrbracket(m)$ over its grammar as:

\[
\begin{align*}
\llbracket n \rrbracket(m) &= n \\
\llbracket x \rrbracket(m) &= m(x) \\
\llbracket E_0 \circledast E_1 \rrbracket(m) &= f_{\circledast}(\llbracket E_0 \rrbracket(m), \llbracket E_1 \rrbracket(m))
\end{align*}
\]

• Here $f_{\circledast}$ is the function that implements $\circledast$, for example:
  • $f_{\circledast}(a, b) = a + b$
def f_binop(op: BinaryOps, left: Scalar, right: Scalar) -> Scalar:
    if op == '+':
        return left + right
    elif op == '-':
        return left - right
    elif op == '*':
        return left * right
    elif op == '/':
        return left // right
    else:
        raise NotImplementedError(f"Unknown operator: {op}")

def evaluate_Expr(E: Expr, m: Memory) -> Scalar:
    if isinstance(E, Scalar):
        return E
    elif isinstance(E, Var):
        return m[E.name]
    elif isinstance(E, BinOp):
        return f_binop(E.op,
                        evaluate_Expr(E.left, m),
                        evaluate_Expr(E.right, m))
Semantics of Boolean Expressions

- Let $\mathbb{B}$ be the set $\{\text{true}, \text{false}\}$
- The semantics of a boolean expression is then $[[B]] : M \rightarrow \mathbb{B}$

$$[[x \otimes n]](m) = f_{\otimes}(m(x), n)$$

which can be expressed in Python as:

```python
def f_cmpop(op: ComparisonOps, left: Scalar, right: Scalar) -> bool:
    if op == '<':
        return left < right
    elif op == '>':
        return left > right
    ...

def evaluate_BoolExpr(B: BoolExpr, m: Memory) -> bool:
    return f_cmpop(B.op, m[B.left.name], B.right)
```
Both $[E]$ and $[B]$ are building blocks for the semantics of other commands.

While they were defined on a single memory store $m$, we’re going to define the semantics for commands on a set of memory states $M$.

- So, $m \in M$, and $M \in \mathcal{P}(M)$
- Where $\mathcal{P}(M)$ denotes the powerset of memory stores

This way, our semantics for commands $[\cdot]_D$ will convert a set of input states to a set of output states.
\[
[C]_\mathcal{P} : \mathcal{P}(M) \rightarrow \mathcal{P}(M)
\]

\[
[\text{skip}]_\mathcal{P}(M) = M
\]

\[
[C_0; C_1]_\mathcal{P}(M) = [C_1]_\mathcal{P}([C_0]_\mathcal{P}(M))
\]

\[
[x := E]_\mathcal{P}(M) = \{m[x \mapsto [E](m)] \mid m \in M\}
\]

\[
[\text{input}(x)]_\mathcal{P}(M) = \{m[x \mapsto n] \mid m \in M, n \in \mathcal{V}\}
\]

- The notation \(m[x \mapsto n]\) is a memory update, it creates a new store identical to \(m\) except that \(x\) is updated to \(n\)
- \(\text{input}(x)\) updates variable \(x\) with a non-deterministic value \(n\)
def evaluate_Cmd(C: Cmd, M: List[Memory]) -> List[Memory]:
    def update_memories(var, value_lambda):
        out = []
        for m in M:
            m_out = dict(m)
            m_out[var] = value_lambda(m)
            out.append(m_out)
        return out

    if isinstance(C, Skip):
        return M
    elif isinstance(C, Program):
        return evaluate_Cmd(C.program, M)
    elif isinstance(C, Assign):
        return update_memories(C.left.name,
                                lambda m: evaluate_Expr(C.right, m))
    elif isinstance(C, Input):
        n = random.randint(0, 100) # could be anything, actually
        return update_memories(C.var.name, lambda _: n)
    elif isinstance(C, Seq):
        return evaluate_Cmd(C.cmd1, evaluate_Cmd(C.cmd0, M))
    ...

- I’ve chosen $M$ to be a list of memories (recall Memory is a Dict[str, int])
Example of using `evaluate_Cmd`

```python
x = Var('x')
y = Var('y')

m1 = {x.name: 3, y.name: 4}
m2 = {x.name: 5, y.name: 6}

M_in = [m1, m2]

M_out = evaluate_Cmd(Assign(x, 7), M_in)

# M_out = [{x: 7, y: 4}, {x: 7, y: 3}]
```
\[
[\text{if}(B)\{C_0\} \text{ else } \{C_1\}]_{\mathcal{P}}(M) = ?
\]

- \(C_0\) (the code executing when \(B\) is true) must only operate on \(m \in M\) where \([B](m)\) evaluates to true.
- \(C_1\) (the code executing when \(B\) is false) must only operate on \(m \in M\) where \([B](m)\) evaluates to false.
- Define a filter function \(\mathcal{F}_B(M)\) such that
  \[
  \mathcal{F}_B(M) = \{ m \in M \mid [B](m) = \text{true} \}
  \]
- Note: \(\mathcal{F}_{\neg B}\) will give us the memories where \(B\) is false.
\[ [\text{if}(B)\{C_0\} \text{ else } \{C_1\}]_{\mathcal{D}}(M) = [C_0]_{\mathcal{D}}(\mathcal{F}_B(M)) \cup [C_1]_{\mathcal{D}}(\mathcal{F}_{\neg B}(M)) \]

- Find stores where \( B \) is true, evaluate \( C_0 \) over them
- Find stores where \( B \) is false, evaluate \( C_1 \) over them
- Combine the two results using \( \cup \)
def filter_memory(B: BoolExpr, M: List[Memory], res = True) -> List[Memory]:
    out = [m for m in M if evaluate_BoolExpr(B, m) == res]
    return list(out)

def evaluate_Cmd(C: Cmd, M: List[Memory]) -> List[Memory]:
    ...

    elif isinstance(C, IfThenElse):
        then_memory = evaluate_Cmd(C.then_, filter_memory(C.cond, M))
        else_memory = evaluate_Cmd(C.else_, filter_memory(C.cond, M,
                                                  res = False))

        return union_memories(then_memory, else_memory)

    ...

def union_memories(M0: List[Memory], M1: List[Memory]) -> List[Memory]:
    # this implementation is, of course, ridiculous

    # convert everything to sets
    M0_set = set([frozenset(m.items()) for m in M0])
    M1_set = set([frozenset(m.items()) for m in M1])

    M_set = M0_set.union(M1_set)

    # convert back to lists of dicts
    return list([dict(m) for m in M_set])
Semantics for While - #1

\[ \text{while}(B)\{C\}\mathcal{P}(M) \]

- \( B \) must be true in \( m \in M \) to execute \( C \) once
  - \( (\llbracket C \rrbracket_{\mathcal{P}} \circ \mathcal{F}_B)(M) \)
- Executing \( C \) twice is similar:
  - \( (\llbracket C \rrbracket_{\mathcal{P}} \circ \mathcal{F}_B)((\llbracket C \rrbracket_{\mathcal{P}} \circ \mathcal{F}_B)(M)) \)
- Let \( F \) be \( \llbracket C \rrbracket_{\mathcal{P}} \circ \mathcal{F}_B \), then execution \( i \) times is represented as
  - \( F^i(M) \), i.e. \( F(F(F(M))) \) for \( i = 3 \)
- If the loop executes \( i \) times and exits, the memory stores are:
  - \( M_i = \mathcal{F}_{\neg B}(F^i(M)) \), because \( B \) must be false when we exit the loop
• Let $M_i = \mathcal{F}_{\neg B}(F^i(M))$ represent executions of the loop body exactly $i$ times, $i \geq 0$

• Then we can define the semantics of those $i$ executions as:

$$\bigcup_{i \geq 0} M_i = \bigcup_{i \geq 0} \mathcal{F}_{\neg B}(F^i(M))$$

$$= \mathcal{F}_{\neg B} \left( \bigcup_{i \geq 0} F^i(M) \right)$$

$$\llbracket \text{while}(B)\{C\} \rrbracket_{\mathcal{P}}(M) = \mathcal{F}_{\neg B} \left( \bigcup_{i \geq 0} ([C]_{\mathcal{P}} \circ \mathcal{F}_B)^i(M) \right)$$

• The semantics of a non-terminating loop are undefined.
def evaluate_Cmd(C: Cmd, M: List[Memory]) -> List[Memory]:
    ...
    elif isinstance(C, While):
        # L0
        out = [m for m in M] # copy all input states

        pre_iter_memories = filter_memory(C.cond, out)
        accum: List[Memory] = []
        while len(pre_iter_memories):
            after_iter_memories = evaluate_Cmd(C.body, pre_iter_memories)
            accum = union_memories(accum, after_iter_memories)

            # only keep memories where the condition is true
            pre_iter_memories = filter_memory(C.cond, after_iter_memories)

        # This computes L0 U (L1 U L2...) and retains only those
        # memory states where the loop has terminated.
        out = filter_memory(C.cond, union_memories(out, accum), res=False)
        return out
Example of While execution

while(x < 7) {
    y := (y + 1);
    x := (x + 1)
}

START  [{x: 4, y: 0}, {x: 5, y: 0}, {x: 8, y: 0}]

pre:    [{x: 4, y: 0}, {x: 5, y: 0}]
after:  [{x: 5, y: 1}, {x: 6, y: 1}]
accum:  [{x: 5, y: 1}, {x: 6, y: 1}]

pre:    [{x: 5, y: 1}, {x: 6, y: 1}]
after:  [{x: 6, y: 2}, {x: 7, y: 2}]
accum:  [{x: 5, y: 1}, {x: 6, y: 1}, {x: 7, y: 2}, {x: 6, y: 2}]

pre:    [{x: 6, y: 2}]
after:  [{x: 7, y: 3}]
accum:  [{x: 7, y: 3}, {x: 6, y: 2}, {x: 5, y: 1}, {x: 6, y: 1}, {x: 7, y: 2}]

END     [{x: 7, y: 3}, {x: 7, y: 2}, {x: 8, y: 0}]
Wrapping up the semantics

- \([ [ C]_{\mathcal{P}} (\emptyset) = \emptyset]\)
  - Starting from an empty set of states leads to an empty set of states

- Key ideas:
  - Grammar \(\rightarrow\) AST
  - AST \(\rightarrow\) Semantics
  - Semantics \(\rightarrow\) Interpreter
Introduction

A Tiny Language and Its Semantics

To be continued ...
Next lecture

- Abstraction, and building an abstract interpreter
- This lecture was based on material from Chapter 3 in Rival and Yi
- You can find the Python code on GitHub
  - This lecture covered tinyast.py and sem.py