Chapter 5: The Witness Reduction Technique
Closure
Closure

A set is closed under an operation
Natural Number

(set)

$(\forall a \in \mathbb{N})(\forall b \in \mathbb{N})[\sigma(a, b) \in \mathbb{N}]$
\( \mathbb{N} \) is closed under multiplication

\[ \text{multiplication}(2,2) = 4 \]
\( \mathbb{N} \) is closed under multiplication

\[ 2 \cdot 2 = 4 \]
\( \mathbb{N} \) is closed under addition

\[ 2 + 5 = 7 \]
$\mathbb{N}$ is NOT closed under subtraction

\[ 2 - 5 = -3 \]
Closure Property

A set of values $\rightarrow$ a class of functions
Function \( \mathbb{N} \rightarrow \mathbb{N} \)
Function Closure

\[(\forall f_1 \in \mathcal{F})(\forall f_2 \in \mathcal{F})[h_{f_1,f_2} \in \mathcal{F}]\]

where \(h_{f_1,f_2}(n) = \sigma(f_1(n), f_2(n))\)

\(\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}\)
\( \mathbb{P} \) is closed under subtraction

\[
(2x^2 + 3x + 4) - (x^2 - 5x - 3) = x^2 + 8x + 7
\]
$\mathbb{P}$ is closed under multiplication

$$(x - 3)(x - 3) = x^2 - 6x + 9$$
\( P \) is closed under division

\[
x \div x^2 = x^{-1}
\]
#P

- Class of functions that can be defined as counting the number of accepting paths of a P time Nondeterministic Turing Machine
- \( \#P = \{ f \mid (\exists \text{NPTM } N)(\forall x)[f(x) = \#acc_N(x)] \} \)
- Each \#P Function is induced by an NPTM
Closure of \#P: Addition

Think about how two separate \#P functions can be closed when added together
### Closure of $\#P$: Addition

1. Let $f_1, f_2 \in \#P$
2. There exists $N_1$ and $N_2$ which are non deterministic machines which on each input $x$,
   \[ f_1(x) = \#\text{acc}_{N_1}(x) \text{ and } f_2(x) = \#\text{acc}_{N_2}(x) \]
3. Consider machine $N$ that makes a non deterministic choice between whether or not it will simulate $N_1$ or $N_2$.
4. $N$’s total number of accepting paths is the total of the number of accepting paths of $N_1$ and $N_2$
5. Think of $N$ as a tree with 2 children, being $N_1$ and $N_2$. 

- Addition:
Closure of \#P: Multiplication
Closure of $\#P$: Multiplication

1. Let $f_1, f_2 \in \#P$
2. There exists $N_1$ and $N_2$ which are non deterministic machines which on each input $x$, $f_1(x) = \#acc_{N_1}(x)$ and $f_2(x) = \#acc_{N_2}(x)$
3. Machine $N$ on input $x$ makes one nondeterministic guess of $N_1$ and one of $N_2$, and accepts if both accept.
What about subtraction? Witness Reduction

- So far, it seems like it is easy to show \#P closures under operations that involve increasing the number of accepting paths. But what about operations that reduce the number of accepting paths?
- Since an NPTM cannot have negative numbers of accepting paths, we will use proper subtraction (denoted $\ominus$) instead.
- Are Natural Numbers closed under proper subtraction?
Approaches to examine closure of proper subtraction
Approach 1:

- Approach like building the theory for NP-completeness
- SAT ∈ P?
Approach 2:

- Characterize the issue of whether $\#P$ is closed under proper subtraction in terms of some class collapse.
Witness Reduction
Witness reduction

- Showing closure for machines that reduced the number of accepting paths (witnesses) is extremely difficult
Witness Reduction: Cont.

- Class UP: (Unambiguous Non deterministic Polynomial Time)- Solvable in P-time on an Unambiguous Turing Machine (≤1 accepting path for each input)
- Given the assumption of some closure property, some collapse occurs (UP = PP):
  - We take some Language in a powerful class, e.g., PP, and take the machine accepting that language and (perhaps after some normalization/manipulation) convert the machine into a #P function.
  - Use the assumed closure operation to create a new #P function
  - Take the new #P function and convert it back into a machine defining a language in a smaller class, for example UP
- Think of it like taking a powerful machine, simplifying its function, and the creating a less powerful machine from that.
An old friend, PP

PP = \{ L \mid \text{there is a PPTM } M \text{ such that for each } x \text{ it holds that } x \in L \iff Pr[M(x) \text{ accepts}] \geq 1/2 \}

So:

- \( x \in L \implies M \text{ accepts } x \text{ with probability } \geq 1/2 \)
- \( x \not\in L \implies M \text{ accepts } x \text{ with probability } < 1/2 \)
A new perspective!

A language \( L \) is in \( \text{PP} \) if there exists a polynomial \( q \) and a polynomial-time computable predicate \( R \) such that, for each \( x \):

\[
x \in L \iff \| \{ y \mid |y| = q(|x|) \land R(x, y) \} \| \geq 2^{q(|x|)-1}
\]

Uhh... what?
A new perspective!

\[ x \in L \iff \| \{ y \mid |y| = q(|x|) \land R(x, y) \} \| \geq 2^{q(|x|)-1} \]
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\[ x \in L \iff \| \{ y \mid |y| = q(|x|) \wedge R(x, y) \} \| \geq 2^{q(|x|)-1} \]
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So what can we find with witness reduction?
Oh... wow

Theorem 5.6 - The following statements are equivalent:

1. \#P is closed under proper subtraction.
2. \#P is closed under every polynomial-time computable operation
3. UP = PP
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1. \#P is closed under proper subtraction.
2. \#P is closed under every polynomial-time computable operation
3. \text{UP} = \text{PP}

Now we'll show that 2 \implies 1, 1 \implies 3, and that 3 \implies 2
Theorem 5.6 (2 ⇒ 1)

The following statements are equivalent:

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2. \( \#P \) is closed under every polynomial-time computable operation
3. \( \text{UP} = \text{PP} \)
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1. \#P is closed under proper subtraction.
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Theorem 5.6 (1 ⇒ 3)
Theorem 5.6 ($1 \Rightarrow 3$)

Since we know $\text{UP} \subseteq \text{PP}$, we only need to show that $\text{PP} \subseteq \text{UP}$.

We are going to do this in two parts:

- $\text{PP} \subseteq \text{coNP}$
- $\text{coNP} \subseteq \text{UP}$
Theorem 5.6 (1 $\Rightarrow$ 3)

Since we know $UP \subseteq PP$, we only need to show that $PP \subseteq UP$.

We are going to do this in two parts:

- $PP \subseteq coNP$
- $coNP \subseteq UP$

1. $\#P$ is closed under proper subtraction
3. $UP = PP$
Theorem 5.6 ($1 \Rightarrow \text{PP} \subseteq \text{coNP}$)

Let $L$ be an arbitrary PP language, so we have:

$L = \{x \mid \|y \mid y = q(|x|) \wedge R(x, y)\| \geq 2^{q(|x|)-1}\}$

We can assume w.l.o.g. that for all $n$, $q(n) \geq 1$. 

1. $\#P$ is closed under proper subtraction

3. $\text{UP} = \text{PP}$
Theorem 5.6 (1 ⇒ PP ⊆ coNP)

$L = \{ x \mid \| \{ y \mid |y| = q(|x|) \land R(x, y) \} \| \geq 2^{q(|x|)-1} \}$

Consider the following NPTM $M$ which on input $x$ does the following:

- Guess each $y$ such that $|y| = q(|x|)$
- Test $R(x, y)$, accept if it evaluates to true. Otherwise, reject

1. #P is closed under proper subtraction
3. UP = PP
Theorem 5.6 \((1 \Rightarrow \text{PP} \subseteq \text{coNP})\)

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\(M\) has the following key property:
Theorem 5.6 ($1 \Rightarrow \text{PP } \subseteq \text{coNP}$)

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$M$ has the following key property:

- $x \in L \Rightarrow M(x)$ has $\geq 2^{q(|x|)-1}$ accepting paths
Theorem 5.6 \((1 \Rightarrow PP \subseteq coNP)\)

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1. \(\#P\) is closed under proper subtraction
3. \(UP = PP\)
Theorem 5.6 \((1 \Rightarrow \text{PP} \subseteq \text{coNP})\)

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So there is a $\#P$ function $f$ with the same property:

- $x \in L \Rightarrow f(x) \geq 2^{q(|x|)-1}$
- $x \notin L \Rightarrow f(x) < 2^{q(|x|)-1}$

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We also have that $g(x) = 2^{q(|x|) - 1} - 1$ is a \#P function as $q$ is a fixed polynomial (and $q(n) \geq 1$)

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Theorem 5.6 (1 ⇒ PP ⊆ coNP)

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Consider \( h(x) = f(x) \oplus g(x) \)

- \( x \in L \Rightarrow h(x) \geq 2^{q(|x|) - 1} \oplus 2^{q(|x|) - 1} - 1 = 1 \)

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Consider \(h(x) = f(x) \Theta g(x)\)

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Consider \(h(x) = f(x) \ominus g(x)\)

- \(x \in L \Rightarrow h(x) \geq 1\)
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1. \#P is closed under proper subtraction

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Consider $h(x) = f(x) \ominus g(x)$

- $x \in L \Rightarrow h(x) \geq 1$
- $x \notin L \Rightarrow h(x) < 2^{q(|x|)−1} \ominus 2^{q(|x|)−1} − 1 = 1$

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Theorem 5.6 (1 ⇒ PP ⊆ coNP)

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We also have that \( g(x) = 2^{q(|x|) - 1} - 1 \) is a \#P function as \( q \) is a fixed polynomial (and \( q(n) \geq 1 \))

Consider \( h(x) = f(x) \ominus g(x) \)

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Consider $h(x) = f(x) \ominus g(x)$

- $x \in L \Rightarrow h(x) \geq 1$
- $x \notin L \Rightarrow 0 \leq h(x) < 1$
Theorem 5.6 (1 ⇒ PP ⊆ coNP)

So there is a #P function f with the same property:

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Theorem 5.6 ($1 \Rightarrow \text{PP} \subseteq \text{coNP}$)

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But remember, we are assuming \(#P\) is closed under proper subtraction, so \(h \in \#P\)!
Theorem 5.6 ($1 \Rightarrow \text{PP} \subseteq \text{coNP}$)

Consider $h(x) = f(x) \ominus g(x)$

- $x \in L \Rightarrow h(x) \geq 1$
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But remember, we are assuming #P is closed under proper subtraction, so $h \in \#P$!

So we know that there is a NPTM $N$ that on every input $x$ has $h(x)$ accepting paths. So:

1. #P is closed under proper subtraction
3. UP = PP
Theorem 5.6 ($1 \Rightarrow \text{PP} \subseteq \text{coNP}$)

Consider $h(x) = f(x) \ominus g(x)$

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But remember, we are assuming $\#P$ is closed under proper subtraction, so $h \in \#P$!

So we know that there is a NPTM $N$ that on every input $x$ has $h(x)$ accepting paths. So:

- $x \in L \Rightarrow \text{acc}_N(x) \geq 1$
Theorem 5.6 ($1 \Rightarrow \text{PP} \subseteq \text{coNP}$)

Consider $h(x) = f(x) \ominus g(x)$

- $x \in L \Rightarrow h(x) \geq 1$
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3. $\text{UP} = \text{PP}$
Theorem 5.6 \((1 \Rightarrow PP \subseteq coNP)\)

Consider \(h(x) = f(x) \Theta g(x)\)

- \(x \in L \Rightarrow h(x) \geq 1\)
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But remember, we are assuming \(\#P\) is closed under proper subtraction, so \(h \in \#P\)!

So we know that there is a NPTM \(N\) that on every input \(x\) has \(h(x)\) accepting paths. So:

- \(x \in L \Rightarrow \#acc_N(x) \geq 1\)
- \(x \in L \Rightarrow \#acc_N(x) = 0\)

So \(L(N) = L\), and we have \(L \in NP\). As \(L\) was an arbitrary PP language we have \(PP \subseteq NP\)

1. \(\#P\) is closed under proper subtraction

3. \(UP = PP\)
Theorem 5.6 ($1 \Rightarrow PP \subseteq coNP$)

Now, as $PP \subseteq NP$:

1. $\#P$ is closed under proper subtraction

3. $UP = PP$
Theorem 5.6 (1 ⇒ PP ⊆ coNP)

Now, as PP ⊆ NP:

$L ∈ PP ⇒ L ∈ NP$

1. #P is closed under proper subtraction

3. UP = PP
**Theorem 5.6** \( 1 \Rightarrow \text{PP} \subseteq \text{coNP} \)

Now, as \( \text{PP} \subseteq \text{NP} \):
\[
\overline{L} \in \text{coPP} \Rightarrow \overline{L} \in \text{coNP}
\]
Theorem 5.6 \((1 \Rightarrow PP \subseteq coNP)\)

Now, as \(PP \subseteq NP:\)

\(\overline{L} \in coPP \Rightarrow \overline{L} \in coNP\)

(remember, \(PP = coPP\))

1. \(\#P\) is closed under proper subtraction

3. \(UP = PP\)
Theorem 5.6 (1 ⇒ PP ⊆ coNP)

Now, as PP ⊆ NP:
\( \overline{L} \in \text{coPP} \Rightarrow \overline{L} \in \text{coNP} \)

(remember, PP = coPP)
Theorem 5.6 (1 ⇒ PP ⊆ coNP)

Now, as PP ⊆ NP:

\( \overline{L} \in PP \Rightarrow \overline{L} \in coNP \)

(remember, PP = coPP)

1. #P is closed under proper subtraction

3. UP = PP
Theorem 5.6 (1 ⇒ PP ⊆ coNP)

Now, as PP ⊆ NP:
\( \overline{L} \in PP \Rightarrow \overline{L} \in \text{coNP} \)
Theorem 5.6 (1 ⇒ PP ⊆ coNP)

Now, as PP ⊆ NP:
\( \overline{L} \in PP \Rightarrow \overline{L} \in \text{coNP} \)
PP ⊆ coNP

1. #P is closed under proper subtraction
3. UP = PP
Theorem 5.6 ($\Rightarrow PP \subseteq coNP$)

Now, as $PP \subseteq NP$:

$L \in PP \Rightarrow \overline{L} \in coNP$

$PP \subseteq coNP$
Theorem 5.6 (1 $\Rightarrow$ 3)

Since we know $\text{UP} \subseteq \text{PP}$, we only need to show that $\text{PP} \subseteq \text{UP}$.

We are going to do this in two parts:

- $\text{PP} \subseteq \text{coNP}$
- $\text{coNP} \subseteq \text{UP}$

1. $\#P$ is closed under proper subtraction
3. $\text{UP} = \text{PP}$
Theorem 5.6 ($1 \Rightarrow 3$)

Since we know $UP \subseteq PP$, we only need to show that $PP \subseteq UP$

We are going to do this in two parts:

- $PP \subseteq coNP$
- $coNP \subseteq UP$
Theorem 5.6 (1 \Rightarrow 3)

Since we know \( \text{UP} \subseteq \text{PP} \), we only need to show that \( \text{PP} \subseteq \text{UP} \).

We are going to do this in two parts:

1. \( \text{P} \) is closed under proper subtraction
2. \( \text{UP} = \text{PP} \)
3. \( \text{PP} \subseteq \text{coNP} \)
4. \( \text{coNP} \subseteq \text{UP} \)
Theorem 5.6 ($1 \Rightarrow \text{coNP} \subseteq \text{UP}$)

Let $L$ be an arbitrary coNP language.

By definition, $L \in \text{coNP} \iff \overline{L} \in \text{NP}$
Theorem 5.6 ($1 \Rightarrow \text{coNP} \subseteq \text{UP}$)

Let $L$ be an arbitrary coNP language.

By definition, $L \in \text{coNP} \iff \overline{L} \in \text{NP}$

So, Let $N$ be an NPTM accepting $\overline{L}$

1. $\#P$ is closed under proper subtraction
2. $\text{UP} = \text{PP}$
Theorem 5.6 (1 ⇔ coNP ⊆ UP)

Let $L$ be an arbitrary coNP language.

By definition, $L \in \text{coNP} \iff \overline{L} \in \text{NP}$

So, let $N$ be an NPTM accepting $\overline{L}$

Notice that $N$ has the following properties on input $x$:

1. $\#P$ is closed under proper subtraction

3. $\text{UP} = \text{PP}$
Theorem 5.6 \((1 \Rightarrow \text{coNP} \subseteq \text{UP})\)

Let \(L\) be an arbitrary coNP language.

By definition, \(L \in \text{coNP} \iff \overline{L} \in \text{NP}\)

So, Let \(N\) be an NPTM accepting \(\overline{L}\)

Notice that \(N\) has the following properties on input \(x\):

- \(x \in L\)
Theorem 5.6 \((1 \Rightarrow \text{coNP} \subseteq \text{UP})\)

Let \(L\) be an arbitrary coNP language.

By definition, \(L \in \text{coNP} \iff \overline{L} \in \text{NP}\)

So, Let \(N\) be an NPTM accepting \(\overline{L}\)

Notice that \(N\) has the following properties on input \(x\):

- \(x \in L \Rightarrow x \notin \overline{L}\)
Theorem 5.6 \((1 \Rightarrow \text{coNP} \subseteq \text{UP})\)

Let \(L\) be an arbitrary coNP language.

By definition, \(L \in \text{coNP} \iff \overline{L} \in \text{NP}\)

So, Let \(N\) be an NPTM accepting \(\overline{L}\)

Notice that \(N\) has the following properties on input \(x\):

- \(x \in L \Rightarrow x \notin \overline{L} \Rightarrow N(x)\) rejects, and has 0 accepting paths
Theorem 5.6 ($1 \Rightarrow \text{coNP} \subseteq \text{UP}$)

Let $L$ be an arbitrary coNP language.

By definition, $L \in \text{coNP} \iff \overline{L} \in \text{NP}$

So, Let $N$ be an NPTM accepting $\overline{L}$

Notice that $N$ has the following properties on input $x$:

- $x \in L \Rightarrow x \notin \overline{L} \Rightarrow N(x)$ rejects, and has 0 accepting paths
- $x \notin L$
Theorem 5.6 ($1 \Rightarrow \text{coNP} \subseteq \text{UP}$)

Let $L$ be an arbitrary coNP language.

By definition, $L \in \text{coNP} \iff \overline{L} \in \text{NP}$

So, Let $N$ be an NPTM accepting $\overline{L}$

Notice that $N$ has the following properties on input $x$:

- $x \in L \Rightarrow x \notin \overline{L} \Rightarrow N(x)$ rejects, and has 0 accepting paths
- $x \notin L \Rightarrow x \in \overline{L}$
Theorem 5.6 ($1 \Rightarrow \text{coNP} \subseteq \text{UP}$)

Let $L$ be an arbitrary coNP language.

By definition, $L \in \text{coNP} \iff \overline{L} \in \text{NP}$

So, Let $N$ be an NPTM accepting $\overline{L}$

Notice that $N$ has the following properties on input $x$:

- $x \in L \Rightarrow x \notin \overline{L} \Rightarrow N(x)$ rejects, and has 0 accepting paths
- $x \notin L \Rightarrow x \in \overline{L} \Rightarrow N(x)$ accepts, and has at least 1 accepting path

1. $\#P$ is closed under proper subtraction

3. $\text{UP} = \text{PP}$
Theorem 5.6 ($1 \Rightarrow \text{coNP} \subseteq \text{UP}$)

Notice that $N$ has the following properties on input $x$:

- $x \in L \Rightarrow x \notin \overline{L} \Rightarrow N(x)$ rejects, and has 0 accepting paths
- $x \notin L \Rightarrow x \in \overline{L} \Rightarrow N(x)$ accepts, and has at least 1 accepting path

1. $\#P$ is closed under proper subtraction

3. $\text{UP} = \text{PP}$
Theorem 5.6 (1 ⇒ coNP ⊆ UP)

Notice that \( N \) has the following properties on input \( x \):

- \( x \in L \Rightarrow x \not\in \overline{L} \Rightarrow N(x) \) rejects, and has 0 accepting paths
- \( x \not\in L \Rightarrow x \in \overline{L} \Rightarrow N(x) \) accepts, and has at least 1 accepting path

1. \#P is closed under proper subtraction
3. UP = PP
Theorem 5.6 \( (1 \Rightarrow \text{coNP} \subseteq \text{UP}) \)

Notice that \( N \) has the following properties on input \( x \):

- \( x \in L \Rightarrow N(x) \) has 0 accepting paths
- \( x \notin L \Rightarrow N(x) \) has at least 1 accepting path

1. \#P is closed under proper subtraction

3. \( \text{UP} = \text{PP} \)
Theorem 5.6 ($1 \Rightarrow \text{coNP} \subseteq \text{UP}$)

Notice that $N$ has the following properties on input $x$:

- $x \in L \Rightarrow N(x)$ has 0 accepting paths
- $x \not\in L \Rightarrow N(x)$ has at least 1 accepting path

So there is a #P function $f$ with the same property:

- $x \in L \Rightarrow f(x) = 0$
- $x \not\in L \Rightarrow f(x) \geq 1$

1. #P is closed under proper subtraction
3. UP = PP
Theorem 5.6 ($1 \Rightarrow \text{coNP} \subseteq \text{UP}$)

So there is a $\#P$ function $f$ with the same property:

- $x \in L \Rightarrow f(x) = 0$
- $x \notin L \Rightarrow f(x) \geq 1$

1. $\#P$ is closed under proper subtraction

3. $\text{UP} = \text{PP}$
Theorem 5.6 \(1 \Rightarrow \text{coNP} \subseteq \text{UP}\)

So there is a \(\#P\) function \(f\) with the same property:

\[
\begin{align*}
\text{if } x \in L & \Rightarrow f(x) = 0 \\
\text{if } x \not\in L & \Rightarrow f(x) \geq 1
\end{align*}
\]

We also have that \(g(x) = 1\) is a \(\#P\) function (Why?)

1. \(\#P\) is closed under proper subtraction
3. \(\text{UP} = \text{PP}\)
Theorem 5.6 \((1 \Rightarrow \text{coNP} \subseteq \text{UP})\)

So there is a \#P function \(f\) with the same property:

* \(x \in L \Rightarrow f(x) = 0\)
* \(x \notin L \Rightarrow f(x) \geq 1\)

We also have that \(g(x) = 1\) is a \#P function

Consider, \(h(x) = g(x) \ominus f(x)\)
Theorem 5.6 \((1 \Rightarrow \text{coNP} \subseteq \text{UP})\)

So there is a \#P function \(f\) with the same property:

- \(x \in L \Rightarrow f(x) = 0\)
- \(x \notin L \Rightarrow f(x) \geq 1\)

We also have that \(g(x) = 1\) is a \#P function

Consider, \(h(x) = g(x) \ominus f(x)\)

- \(x \in L \Rightarrow h(x) = 1 \ominus 0 = 1\)

1. \#P is closed under proper subtraction

3. \(\text{UP} = \text{PP}\)
Theorem 5.6 ($1 \Rightarrow \text{coNP} \subseteq \text{UP}$)

So there is a #P function $f$ with the same property:

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Consider, $h(x) = g(x) \ominus f(x)$

- $x \in L \Rightarrow h(x) = 1 \ominus 0 = 1$

1. #P is closed under proper subtraction
3. UP = PP
Theorem 5.6 \((1 \Rightarrow \text{coNP} \subseteq \text{UP})\)

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- \( x \in L \Rightarrow h(x) = 1 \ominus 0 = 1 \)
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- \(x \in L \Rightarrow h(x) = 1 \ominus 0 = 1\)

1. \#P is closed under proper subtraction
3. \(\text{UP} = \text{PP}\)
Theorem 5.6 (1 ⇒ coNP ⊆ UP)

So there is a #P function f with the same property:

- \( x \in L \Rightarrow f(x) = 0 \)
- \( x \notin L \Rightarrow f(x) \geq 1 \)

We also have that \( g(x) = 1 \) is a #P function

Consider, \( h(x) = g(x) \Theta f(x) \)

- \( x \in L \Rightarrow h(x) = 1 \Theta 0 = 1 \)
- \( x \notin L \Rightarrow h(x) \leq 1 \Theta 1 = 0 \)
Theorem 5.6 (1 ⇒ coNP ⊆ UP)

So there is a #P function f with the same property:

- $x \in L \Rightarrow f(x) = 0$
- $x \not\in L \Rightarrow f(x) \geq 1$

We also have that $g(x) = 1$ is a #P function

Consider, $h(x) = g(x) \ominus f(x)$

- $x \in L \Rightarrow h(x) = 1 \ominus 0 = 1$
- $x \not\in L \Rightarrow h(x) \leq 1 \ominus 1 = 0$

1. #P is closed under proper subtraction
3. UP = PP
Theorem 5.6 \((1 \Rightarrow \text{coNP} \subseteq \text{UP})\)

So there is a \#P function \(f\) with the same property:

- \(x \in L \Rightarrow f(x) = 0\)
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We also have that \(g(x) = 1\) is a \#P function

Consider, \(h(x) = g(x) \ominus f(x)\)

- \(x \in L \Rightarrow h(x) = 1 \ominus 0 = 1\)
- \(x \notin L \Rightarrow h(x) \leq 1 \ominus 1 = 0\)
Theorem 5.6 ($1 \Rightarrow \text{coNP} \subseteq \text{UP}$)

So there is a $\#P$ function $f$ with the same property:

- $x \in L \Rightarrow f(x) = 0$
- $x \not\in L \Rightarrow f(x) \geq 1$

We also have that $g(x) = 1$ is a $\#P$ function

Consider, $h(x) = g(x) \ominus f(x)$

- $x \in L \Rightarrow h(x) = 1 \ominus 0 = 1$
- $x \not\in L \Rightarrow h(x) \leq 1 \ominus 1 = 0$

1. $\#P$ is closed under proper subtraction

3. $\text{UP} = \text{PP}$
Theorem 5.6 \((1 \Rightarrow \text{coNP} \subseteq \text{UP})\)

So there is a \#P function \(f\) with the same property:

- \(x \in L \Rightarrow f(x) = 0\)
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We also have that \(g(x) = 1\) is a \#P function

Consider, \(h(x) = g(x) \ominus f(x)\)

- \(x \in L \Rightarrow h(x) = 1 \ominus 0 = 1\)
- \(x \notin L \Rightarrow h(x) \leq 1 \ominus 1 = 0\)

1. \#P is closed under proper subtraction
3. \(\text{UP} = \text{PP}\)
Theorem 5.6 (1 ⇒ coNP ⊆ UP)

So there is a #P function \( f \) with the same property:

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We also have that \( g(x) = 1 \) is a #P function

Consider, \( h(x) = g(x) \ominus f(x) \)

- \( x \in L \Rightarrow h(x) = 1 \ominus 0 = 1 \)
- \( x \notin L \Rightarrow h(x) \leq 1 \ominus 1 = 0 \)
Theorem 5.6 ($1 \Rightarrow \text{coNP} \subseteq \text{UP}$)

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Consider, $h(x) = g(x) \ominus f(x)$

- $x \in L \Rightarrow h(x) = 1$
- $x \notin L \Rightarrow h(x) \leq 1 \ominus 1 = 0$
Theorem 5.6 \( (1 \Rightarrow \text{coNP} \subseteq \text{UP}) \)

So there is a \#P function \( f \) with the same property:

- \( x \in L \Rightarrow f(x) = 0 \)
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We also have that \( g(x) = 1 \) is a \#P function

Consider, \( h(x) = g(x) \ominus f(x) \)

- \( x \in L \Rightarrow h(x) = 1 \)
- \( x \notin L \Rightarrow h(x) \leq 1 \ominus 1 = 0 \)
Theorem 5.6 ($1 \Rightarrow \text{coNP} \subseteq \text{UP}$)

So there is a \#P function $f$ with the same property:

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Consider, $h(x) = g(x) \ominus f(x)$

- $x \in L \Rightarrow h(x) = 1$
- $x \notin L \Rightarrow h(x) \leq 0$
Theorem 5.6 ($1 \Rightarrow \text{coNP} \subseteq \text{UP}$)

So there is a \#P function $f$ with the same property:

- $x \in L \Rightarrow f(x) = 0$
- $x \notin L \Rightarrow f(x) \geq 1$

We also have that $g(x) = 1$ is a \#P function

Consider, $h(x) = g(x) \ominus f(x)$

- $x \in L \Rightarrow h(x) = 1$
- $x \notin L \Rightarrow 0 \leq h(x) \leq 0$

1. \#P is closed under proper subtraction

3. UP = PP
Theorem 5.6 \((1 \Rightarrow \text{coNP} \subseteq \text{UP})\)

So there is a \#P function \(f\) with the same property:

- \(x \in L \Rightarrow f(x) = 0\)
- \(x \notin L \Rightarrow f(x) \geq 1\)

We also have that \(g(x) = 1\) is a \#P function.

Consider, \(h(x) = g(x) \ominus f(x)\)

- \(x \in L \Rightarrow h(x) = 1\)
- \(x \notin L \Rightarrow h(x) = 0\)

1. \#P is closed under proper subtraction

3. \(\text{UP} = \text{PP}\)
Consider,

\[ h(x) = g(x) \ominus f(x) \]

- \( x \in L \Rightarrow h(x) = 1 \)
- \( x \notin L \Rightarrow h(x) = 0 \)

Theorem 5.6 (1 \( \Rightarrow \) \text{coNP} \subseteq \text{UP})

1. \#P is closed under proper subtraction

3. \text{UP} = \text{PP}
Consider, $h(x) = g(x) \ominus f(x)$

- $x \in L \Rightarrow h(x) = 1$
- $x \notin L \Rightarrow h(x) = 0$

But remember, we are assuming $\#P$ is closed under proper subtraction, so $h \in \#P$!
Consider, $h(x) = g(x) \ominus f(x)$

- $x \in L \Rightarrow h(x) = 1$
- $x \notin L \Rightarrow h(x) = 0$

But remember, we are assuming $\#P$ is closed under proper subtraction, so $h \in \#P$!

So we know that there is a NPTM $N$ that on every input $x$ has $h(x)$ accepting paths. So:
Consider, $h(x) = g(x) \ominus f(x)$

- $x \in L \Rightarrow h(x) = 1$
- $x \notin L \Rightarrow h(x) = 0$

But remember, we are assuming $\#P$ is closed under proper subtraction, so $h \in \#P$!

So we know that there is a NPTM $N$ that on every input $x$ has $h(x)$ accepting paths. So:

- $x \in L \Rightarrow N(x)$ has 1 accepting path

1. $\#P$ is closed under proper subtraction
3. $\text{UP} = \text{PP}$
Consider, \( h(x) = g(x) \oplus f(x) \)

- \( x \in L \Rightarrow h(x) = 1 \)
- \( x \notin L \Rightarrow h(x) = 0 \)

But remember, we are assuming \#P is closed under proper subtraction, so \( h \in \#P \)!

So we know that there is a NPTM \( N \) that on every input \( x \) has \( h(x) \) accepting paths. So:

- \( x \in L \Rightarrow N(x) \) has 1 accepting path
- \( x \notin L \Rightarrow N(x) \) has 0 accepting paths

1. \( \#P \) is closed under proper subtraction

3. \( \text{UP} = \text{PP} \)

**Theorem 5.6** \((1 \Rightarrow \text{coNP} \subseteq \text{UP})\)
Consider, $h(x) = g(x) \ominus f(x)$

- $x \in L \Rightarrow h(x) = 1$
- $x \notin L \Rightarrow h(x) = 0$

But remember, we are assuming \#P is closed under proper subtraction, so $h \in \#P$!

So we know that there is a NPTM $N$ that on every input $x$ has $h(x)$ accepting paths. So:

- $x \in L \Rightarrow N(x)$ has 1 accepting path
- $x \notin L \Rightarrow N(x)$ has 0 accepting paths

So $L(N) = L$, and $N$ shows that $L \in \text{UP}$. As $L$ was an arbitrary coNP language, we have $\text{coNP} \subseteq \text{UP}$.

1. \#P is closed under proper subtraction

3. $\text{UP} = \text{PP}$
Consider, $h(x) = g(x) \ominus f(x)$

- $x \in L \Rightarrow h(x) = 1$
- $x \notin L \Rightarrow h(x) = 0$

But remember, we are assuming \#P is closed under proper subtraction, so $h \in \#P$!

So we know that there is a NPTM $N$ that on every input $x$ has $h(x)$ accepting paths. So:

- $x \in L \Rightarrow N(x)$ has 1 accepting path
- $x \notin L \Rightarrow N(x)$ has 0 accepting paths

So $L(N) = L$, and $N$ shows that $L \in UP$. As $L$ was an arbitrary coNP language, we have $\text{coNP} \subseteq UP$.
Theorem 5.6 (1 ⇒ 3)

Since we know $UP \subseteq PP$, we only need to show that $PP \subseteq UP$

We are going to do this in two parts:

- $PP \subseteq coNP$
- $coNP \subseteq UP$
Theorem 5.6 (1 $\Rightarrow$ 3)

Since we know UP $\subseteq$ PP, we only need to show that PP $\subseteq$ UP.

We are going to do this in two parts:

- PP $\subseteq$ coNP
- coNP $\subseteq$ UP

1. #P is closed under proper subtraction
3. UP = PP
Theorem 5.6 ($1 \Rightarrow 3$)

Since we know $\text{UP} \subseteq \text{PP}$, we only need to show that $\text{PP} \subseteq \text{UP}$.

We are going to do this in two parts:

- $\text{PP} \subseteq \text{coNP}$
- $\text{coNP} \subseteq \text{UP}$
Theorem 5.6 ($1 \Rightarrow 3$)

Since we know $\text{UP} \subseteq \text{PP}$, we only need to show that $\text{PP} \subseteq \text{UP}$

We are going to do this in two parts:

- $\text{PP} \subseteq \text{coNP}, \text{coNP} \subseteq \text{UP}$

1. $\#P$ is closed under proper subtraction

3. $\text{UP} = \text{PP}$
Theorem 5.6 \((1 \Rightarrow 3)\)

Since we know \(\text{UP} \subseteq \text{PP}\), we only need to show that \(\text{PP} \subseteq \text{UP}\).

We are going to do this in two parts:

1. \#P is closed under proper subtraction
2. \(\text{UP} = \text{PP}\)

- \(\text{PP} \subseteq \text{coNP} \subseteq \text{UP}\)
Theorem 5.6 \((1 \Rightarrow 3)\)

Since we know \(\text{UP} \subseteq \text{PP}\), we only need to show that \(\text{PP} \subseteq \text{UP}\).

We are going to do this in two parts:

- \(\text{PP} \subseteq \text{UP}\)
Theorem 5.6 ($1 \Rightarrow 3$)

Since we know $UP \subseteq PP$, we only need to show that $PP \subseteq UP$

We are going to do this in two parts:

- $PP \subseteq UP$

1. $\#P$ is closed under proper subtraction

3. $UP = PP$
Theorem 5.6 ($1 \Rightarrow 3$)

All together, we have just shown:

- $\text{UP} = \text{PP}$
The following statements are equivalent:

1. \( \#P \) is closed under proper subtraction.
2. \( \#P \) is closed under every polynomial-time computable operation
3. \( UP = PP \)
The following statements are equivalent:

1. \( \text{#P is closed under proper subtraction.} \)
2. \( \text{#P is closed under every polynomial-time computable operation} \)
3. \( \text{UP = PP} \)
Theorem 5.6 (3 ⇒ 2)

Let \( op : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) be an arbitrary polynomial-time computable operation.

Let \( f \) and \( g \) be arbitrary \#P functions.
Theorem 5.6 (3 $\Rightarrow$ 2)

Let $op : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be an arbitrary polynomial-time computable operation.

Let $f$ and $g$ be arbitrary #P functions.

Our goal now is to show that $h(x) = op(f(x), g(x))$ is also a #P function.
Theorem 5.6 (3 ⇒ 2)

Let $op : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be an arbitrary polynomial-time computable operation.

Let $f$ and $g$ be arbitrary #P functions.

Our goal now is to show that $h(x) = op(f(x), g(x))$ is also a #P function.

Consider the following set:
Theorem 5.6 ($3 \Rightarrow 2$)

Let $op : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be an arbitrary polynomial-time computable operation.

Let $f$ and $g$ be arbitrary $\#P$ functions.

Our goal now is to show that $h(x) = op(f(x), g(x))$ is also a $\#P$ function.

Consider the following set:

- $B_f = \{ \langle x, n \rangle \mid f(x) \geq n \}$

2. $\#P$ is closed under every polynomial-time computable operation.

3. $UP = PP$
Theorem 5.6 (3 ⇒ 2)

Consider the following set:

- \( B_f = \{ \langle x, n \rangle \mid f(x) \geq n \} \)

3. \( \text{UP} = \text{PP} \)

2. \( \text{#P} \) is closed under every polynomial-time computable operation
Theorem 5.6 (3 $\Rightarrow$ 2)

Consider the following set:

- $B_f = \{ \langle x, n \rangle | f(x) \geq n \}$

Is $B_f \in \text{PP}$?
$B_f \in \text{PP}$

We are going to approach this construction using the equivalence of PP and PP$_{\text{path}}$. 

$B_f = \{<x,n> \mid f(x) \geq n\}$
\[ B_f \subseteq PP \]

We are going to approach this construction using the equivalence of \( PP \) and \( PP_{\text{path}} \).

Let \( N \) be an NPTM corresponding to \( f \).

Let \( m \) be the total number of paths of \( N(x) \), and \( k = f(x) \) be the number of accepting paths of \( N(x) \).
\[ B_f = \{ \langle x, n \rangle \mid f(x) \geq n \} \]

\[ B_f \in \text{PP} \]

We are going to approach this construction using the equivalence of \( \text{PP} \) and \( \text{PP}_{\text{path}} \)

Let \( N \) be an NPTM corresponding to \( f \)

Let \( m \) be the total number of paths of \( N(x) \), and \( k = f(x) \) be the number of accepting paths of \( N(x) \)

Our goal is to construct a PPTM \( M \) such that:

\[ f(x) \geq n \iff \text{number of accepting paths of } M(x) \geq \text{number of rejecting paths of } M(x) \]
$B_f \in \text{PP}$

We are going to approach this construction using the equivalence of \text{PP} and \text{PP}_{\text{path}}.

Let $N$ be an NPTM corresponding to $f$.

Let $m$ be the total number of paths of $N(x)$, and $k = f(x)$ be the number of accepting paths of $N(x)$.

Our goal is to construct a PPTM $M$ such that:

$f(x) \geq n \iff \text{number of accepting paths of } M(x) \geq \text{number of rejecting paths of } M(x)$.
We are going to approach this construction using the equivalence of $\text{PP}$ and $\text{PP}_{\text{path}}$.

Let $N$ be an NPTM corresponding to $f$.

Let $m$ be the total number of paths of $N(x)$, and $k = f(x)$ be the number of accepting paths of $N(x)$.

Our goal is to construct a PPTM $M$ such that:

$$f(x) \geq n \iff \#\text{acc}_M(x) \geq \text{number of rejecting paths of } M(x)$$
We are going to approach this construction using the equivalence of PP and PP\textsubscript{path}.

Let $N$ be an NPTM corresponding to $f$.

Let $m$ be the total number of paths of $N(x)$, and $k = f(x)$ be the number of accepting paths of $N(x)$.

Our goal is to construct a PPTM $M$ such that:

$f(x) \geq n \iff \#\text{acc}_M(x) \geq \text{number of rejecting paths of } M(x)$

$$B_f = \{ \langle x, n \rangle \mid f(x) \geq n \}$$
We are going to approach this construction using the equivalence of PP and \( \text{PP}_{\text{path}} \).

Let \( \mathcal{N} \) be an NPTM corresponding to \( f \).

Let \( m \) be the total number of paths of \( \mathcal{N}(x) \), and \( k = f(x) \) be the number of accepting paths of \( \mathcal{N}(x) \).

Our goal is to construct a PPTM \( \mathcal{M} \) such that:

\[
f(x) \geq n \iff \#\text{acc}_\mathcal{M}(x) \geq \text{number of rejecting paths of } \mathcal{M}(x)
\]
We are going to approach this construction using the equivalence of PP and PP\_path

Let $N$ be an NPTM corresponding to $f$

Let $m$ be the total number of paths of $N(x)$, and $k = f(x)$ be the number of accepting paths of $N(x)$

Our goal is to construct a PPTM $M$ such that:

\[ f(x) \geq n \iff \text{acc}_M(x) \geq \text{rej}_M(x) \]
We are going to approach this construction using the equivalence of PP and PP\textsubscript{path}.

Let $N$ be an NPTM corresponding to $f$.

Let $m$ be the total number of paths of $N(x)$, and $k = f(x)$ be the number of accepting paths of $N(x)$.

Our goal is to construct a PPTM $M$ such that:

$$f(x) \geq n \iff \#acc_M(x) \geq \#rej_M(x)$$
\[ B_f = \{ \langle x, n \rangle \mid f(x) \geq n \} \]

\( B_f \in \text{PP} \)

We are going to approach this construction using the equivalence of PP and PPTM \( P_{\text{path}} \)

Let \( N \) be an NPTM corresponding to \( f \)

Let \( m \) be the total number of paths of \( N(x) \), and \( k = f(x) \) be the number of accepting paths of \( N(x) \)

Our goal is to construct a PPTM \( M \) such that:

\[ f(x) \geq n \iff \#acc_M(x) \geq \#rej_M(x) \]
We are going to approach this construction using the equivalence of PP and \( \text{PP}_{\text{path}} \).

Let \( N \) be an NPTM corresponding to \( f \).

Let \( m \) be the total number of paths of \( N(x) \), and \( k = f(x) \) be the number of accepting paths of \( N(x) \).

Our goal is to construct a PPTM \( M \) such that:

\[ k \geq n \iff \#\text{acc}_M(x) \geq \#\text{rej}_M(x) \]
We are going to approach this construction using the equivalence of \( \text{PP} \) and \( \text{PP}_{\text{path}} \).

Let \( N \) be an NPTM corresponding to \( f \).

Let \( m \) be the total number of paths of \( N(x) \), and \( k = f(x) \) be the number of accepting paths of \( N(x) \).

Our goal is to construct a PPTM \( M \) such that:

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k \geq n \iff \#\text{acc}_M(x) \geq \#\text{rej}_M(x)
\]
Let $N$ be an NPTM corresponding to $f$

Our goal is to construct a PPTM $M$ such that:

\[ k \geq n \iff \text{acc}_M(x) \geq \text{rej}_M(x) \]
Let $N$ be an NPTM corresponding to $f$

Our goal is to construct a PPTM $M$ such that:

$$k \geq n \iff \#acc_M(x) \geq \#rej_M(x)$$

If we have $M$ simply simulate $N$ we get:

$$k \geq n \iff \#acc_M(x) \geq \#rej_M(x)$$
Let $N$ be an NPTM corresponding to $f$

Our goal is to construct a PPTM $M$ such that:

$$k \geq n \iff \#acc_M(x) \geq \#rej_M(x)$$

If we have $M$ simply simulate $N$ we get:

$$k \geq n \iff \#acc_M(x) \geq \#rej_M(x)$$

$$B_f = \{ \langle x, n \rangle \mid f(x) \geq n \}$$

$m$: total # of paths of $N(x)$

$k$: # of acc. paths of $N(x)$
Let $N$ be an NPTM corresponding to $f$

Our goal is to construct a PPTM $M$ such that:

$$k \geq n \iff \#acc_M(x) \geq \#rej_M(x)$$

If we have $M$ simply simulate $N$ we get:

$$k \geq n \iff \#acc_N(x) \geq \#rej_M(x)$$

$$B_f = \{ (x, n) \mid f(x) \geq n \}$$

$m$: total # of paths of $N(x)$

$k$: # of acc. paths of $N(x)$
Let $N$ be an NPTM corresponding to $f$

Our goal is to construct a PPTM $M$ such that:

$$k \geq n \iff \#acc_M(x) \geq \#rej_M(x)$$

If we have $M$ simply simulate $N$ we get:

$$k \geq n \iff k \geq \#rej_M(x)$$

$$B_f = \{ (x, n) \mid f(x) \geq n \}$$

$m$: total # of paths of $N(x)$

$k$: # of acc. paths of $N(x)$
Let \( N \) be an NPTM corresponding to \( f \).

Our goal is to construct a PPTM \( M \) such that:

\[
k \geq n \iff \text{acc}_M(x) \geq \text{rej}_M(x)
\]

If we have \( M \) simply simulate \( N \) we get:

\[
k \geq n \iff k \geq \#\text{rej}_M(x)
\]

\[
B_f = \{ (x,n) \mid f(x) \geq n \}
\]

\( m \): total \# of paths of \( N(x) \)

\( k \): \# of acc. paths of \( N(x) \)
Let $N$ be an NPTM corresponding to $f$

Our goal is to construct a PPTM $M$ such that:

\[ k \geq n \Leftrightarrow \text{acc}_M(x) \geq \text{rej}_M(x) \]

If we have $M$ simply simulate $N$ we get:

\[ k \geq n \Leftrightarrow k \geq \#\text{rej}_M(x) \]

\[ B_f = \{ (x, n) \mid f(x) \geq n \} \]

$m$: total # of paths of $N(x)$

$k$: # of acc. paths of $N(x)$
Let $N$ be an NPTM corresponding to $f$

Our goal is to construct a PPTM $M$ such that:

$$k \geq n \iff \#acc_M(x) \geq \#rej_M(x)$$

If we have $M$ simply simulate $N$ we get:

$$k \geq n \iff k \geq \#rej_N(x)$$
Let $N$ be an NPTM corresponding to $f$

Our goal is to construct a PPTM $M$ such that:

$$k \geq n \iff \text{acc}_M(x) \geq \text{rej}_M(x)$$

If we have $M$ simply simulate $N$ we get:

$$k \geq n \iff k \geq m - k$$

$$B_f = \{ (x, n) \mid f(x) \geq n \}$$

$m$: total # of paths of $N(x)$

$k$: # of acc. paths of $N(x)$
Let $N$ be an NPTM corresponding to $f$

Our goal is to construct a PPTM $M$ such that:

$$k \geq n \iff \#acc_M(x) \geq \#rej_M(x)$$

If we have $M$ simply simulate $N$ we get:

$$k \geq n \iff k \geq m - k$$

$B_f = \{ (x, n) \mid f(x) \geq n \}$

$m$: total # of paths of $N(x)$

$k$: # of acc. paths of $N(x)$
Let $N$ be an NPTM corresponding to $f$.

Our goal is to construct a PPTM $M$ such that:

$$k \geq n \iff \#acc_M(x) \geq \#rej_M(x)$$

If we have $M$ simply simulate $N$ we get:

$$k \geq n \iff k \geq m - k$$

$$B_f = \{ (x, n) \mid f(x) \geq n \}$$

$m$: total # of paths of $N(x)$

$k$: # of acc. paths of $N(x)$
Let $N$ be an NPTM corresponding to $f$

Our goal is to construct a PPTM $M$ such that:

$$k \geq n \iff \#acc_M(x) \geq \#rej_M(x)$$

If we have $M$ simply simulate $N$ we get:

$$k \geq n \iff k \geq m - k$$

$$B_f = \{ (x, n) \mid f(x) \geq n \}$$

$m$: total # of paths of $N(x)$

$k$: # of acc. paths of $N(x)$
$B_f \in \text{PP}$

Let $N$ be an NPTM corresponding to $f$

Our goal is to construct a PPTM $M$ such that:

$k \geq n \iff \text{acc}_M(x) \geq \text{rej}_M(x)$

If we have $M$ simply simulate $N$ we get:

$k \geq n \iff k \geq m - k + (k + n)$
Let $N$ be an NPTM corresponding to $f$

Our goal is to construct a PPTM $M$ such that:

$$k \geq n \iff \#acc_M(x) \geq \#rej_M(x)$$

If we have $M$ simply simulate $N$ we get:

$$k \geq n \iff k \geq m - k + (k + n)$$

$$k \geq n \iff k \geq m + n$$

$$B_f = \{ (x, n) \mid f(x) \geq n \}$$

$m$: total # of paths of $N(x)$

$k$: # of acc. paths of $N(x)$
Let $N$ be an NPTM corresponding to $f$. Our goal is to construct a PPTM $M$ such that:

$$k \geq n \iff \text{acc}_M(x) \geq \text{rej}_M(x)$$

If we have $M$ simply simulate $N$ we get:

$$k \geq n \iff m + k \geq m - k + (k + n)$$

$$k \geq n \iff m + k \geq m + n$$

**Notes:**

- $B_f = \{ (x, n) \mid f(x) \geq n \}$
- $m$: total # of paths of $N(x)$
- $k$: # of acc. paths of $N(x)$
Let $N$ be an NPTM corresponding to $f$

Our goal is to construct a PPTM $M$ such that:

$$k \geq n \iff \#acc_M(x) \geq \#rej_M(x)$$

If we have $M$ simply simulate $N$ we get:

$$k \geq n \iff m + k \geq m - k + (k + n)$$

$$k \geq n \iff m + k \geq m + n$$

$$B_f = \{ (x, n) | f(x) \geq n \}$$

$m$: total # of paths of $N(x)$

$k$: # of acc. paths of $N(x)$
Let $N$ be an NPTM corresponding to $f$

Our goal is to construct a PPTM $M$ such that:

\[ k \geq n \iff \text{acc}_M(x) \geq \text{rej}_M(x) \]

If we have $M$ simply simulate $N$ we get:

\[ k \geq n \iff m + k \geq m - k + (k + n) \]

\[ k \geq n \iff m + k \geq m + n \]
$B_f \in \text{PP}$

So with our new approach we want:

$k \geq n \Leftrightarrow \#\text{acc}_M(x) \geq \#\text{rej}_M(x)$

$k \geq n \Leftrightarrow m + k \geq m - k + (k + n)$

$B_f = \{(x, n) | f(x) \geq n\}$

$m$: total # of paths of $N(x)$

$k$: # of acc. paths of $N(x)$
So with our new approach we want:

\[ k \geq n \iff \text{acc}_M(x) \geq \text{rej}_M(x) \]

\[ k \geq n \iff m + k \geq m - k + (k + n) \]

How can we accomplish this?
\( B_f \in \text{PP} \)

So with our new approach we want:

\[ k \geq n \iff \#acc_M(x) \geq \#rej_M(x) \]

\[ k \geq n \iff m + k \geq m - k + (k + n) \]

How can we accomplish this?

\[ \#acc_M(x) = k \]

\[ \#rej_M(x) = m - k \]
$B_f \in \text{PP}$

So with our new approach we want:

$k \geq n \iff \#\text{acc}_M(x) \geq \#\text{rej}_M(x)$

$k \geq n \iff m + k \geq m - k + (k + n)$

How can we accomplish this?

$$\#\text{acc}_M(x) = \begin{cases} k & \text{original} \\ m & \text{accepting} \end{cases}$$

$$\#\text{rej}_M(x) = \begin{cases} m - k & \text{original} \\ \text{no paths} & \text{accepting} \end{cases}$$

$B_f = \{(x, n) \mid f(x) \geq n\}$

$m$: total # of paths of $N(x)$

$k$: # of acc. paths of $N(x)$
So with our new approach we want:

\[ k \geq n \iff \#acc_M(x) \geq \#rej_M(x) \]

\[ k \geq n \iff m + k \geq m - k + (k + n) \]

How can we accomplish this?

\[ \#acc_M(x) = \]

\[ \#rej_M(x) = m - k \]

\[ B_f = \{ (x, n) \mid f(x) \geq n \} \]

\( m \): total # of paths of \( N(x) \)

\( k \): # of acc. paths of \( N(x) \)
\( B_f \in \text{PP} \)

So with our new approach we want:

\[ k \geq n \iff \#\text{acc}_M(x) \geq \#\text{rej}_M(x) \]

\[ k \geq n \iff m + k \geq m - k + (k + n) \]

How can we accomplish this?

\[ \#\text{acc}_M(x) = \begin{cases} k & \text{original} \\ m & \text{accepting} \end{cases} \]

\[ \#\text{rej}_M(x) = \begin{cases} m - k & \text{flipped} \\ \sim k & \text{total} \end{cases} \]

\[ B_f = \{ \langle x, n \rangle \mid f(x) \geq n \} \]

\( m \): total # of paths of \( N(x) \)

\( k \): # of acc. paths of \( N(x) \)
$B_f \in \text{PP}$

So with our new approach we want:

$k \geq n \iff \#\text{acc}_M(x) \geq \#\text{rej}_M(x)$

$k \geq n \iff m + k \geq m - k + (k + n)$

How can we accomplish this?

$$\#\text{acc}_M(x) = k \quad m \quad \sim 0$$

$$\#\text{rej}_M(x) = m - k \quad k \quad n$$
Theorem 5.6 (3 ⇒ 2)

Consider the following set:

- \( B_f = \{\langle x, n \rangle | f(x) \geq n \} \)

Is \( B_f \in \text{PP} \)?

3. \( \text{UP} = \text{PP} \)

2. \#P is closed under every polynomial-time computable operation
Theorem 5.6 ($3 \Rightarrow 2$)

Consider the following set:

- $B_f = \{ \langle x, n \rangle | f(x) \geq n \}$

Is $B_f \in \text{PP}$? Yes!
Theorem 5.6 (3 ⇒ 2)

Consider the following set:

- \( B_f = \{ \langle x, n \rangle \mid f(x) \geq n \} \in \text{PP} \)

2. \#P is closed under every polynomial-time computable operation

3. UP = PP
Theorem 5.6 ($3 \Rightarrow 2$)

Consider the following set:

- $B_f = \{ \langle x, n \rangle \mid f(x) \geq n \} \in \text{PP}$

3. UP = PP

2. $\#P$ is closed under every polynomial-time computable operation
Consider the following sets:

- \( B_f = \{ \langle x, n \rangle \mid f(x) \geq n \} \in \text{PP} \)
- \( B_g = \{ \langle x, n \rangle \mid g(x) \geq n \} \)

Theorem 5.6 (\(3 \Rightarrow 2\))

3. \( \text{UP} = \text{PP} \)

2. \#P is closed under every polynomial-time computable operation
Consider the following sets:

- \( B_f = \{ \langle x, n \rangle \mid f(x) \geq n \} \in PP \)
- \( B_g = \{ \langle x, n \rangle \mid g(x) \geq n \} \in PP \)

**Theorem 5.6** (3 \( \Rightarrow \) 2)

3. \( UP = PP \)

2. \( \#P \) is closed under every polynomial-time computable operation
Theorem 5.6 (3 ⇒ 2)

Consider the following sets:

- $B_f = \{ \langle x, n \rangle \mid f(x) \geq n \} \in \mathrm{PP}$
- $B_g = \{ \langle x, n \rangle \mid g(x) \geq n \} \in \mathrm{PP}$

3. $\mathrm{UP} = \mathrm{PP}$

2. $\#P$ is closed under every polynomial-time computable operation
Theorem 5.6 ($3 \Rightarrow 2$)

- $B_f = \{ \langle x, n \rangle | f(x) \geq n \} \in \text{PP}$
- $B_g = \{ \langle x, n \rangle | g(x) \geq n \} \in \text{PP}$

3. UP = PP

2. \#P is closed under every polynomial-time computable operation
Theorem 5.6 (3 \implies 2)

- \( B_f = \{ \langle x, n \rangle \mid f(x) \geq n \} \in \text{PP} \)
- \( B_g = \{ \langle x, n \rangle \mid g(x) \geq n \} \in \text{PP} \)

Now consider:

\[
V = \{ \langle x, n_1, n_2 \rangle \mid \langle x, n_1 \rangle \in B_f \land \langle x, n_1 + 1 \rangle \notin B_f \land \langle x, n_2 \rangle \in B_g \land \langle x, n_2 + 1 \rangle \notin B_g \}
\]
Theorem 5.6 ($3 \Rightarrow 2$)

- $B_f = \{ \langle x, n \rangle | f(x) \geq n \} \in \text{PP}$
- $B_g = \{ \langle x, n \rangle | g(x) \geq n \} \in \text{PP}$

Now consider:

$V = \{ \langle x, n_1, n_2 \rangle | \langle x, n_1 \rangle \in B_f \land \langle x, n_1 + 1 \rangle \notin B_f \land$

$\langle x, n_2 \rangle \in B_g \land \langle x, n_2 + 1 \rangle \notin B_g \}$

3. UP = PP

2. #P is closed under every polynomial-time computable operation
Theorem 5.6 (3 ⇒ 2)

- \( B_f = \{ \langle x, n \rangle | f(x) \geq n \} \in \text{PP} \)
- \( B_g = \{ \langle x, n \rangle | g(x) \geq n \} \in \text{PP} \)

Now consider:

\[
V = \{ \langle x, n_1, n_2 \rangle | f(x) \geq n_1 \land \langle x, n_1 + 1 \rangle \notin B_f \land \\
\langle x, n_2 \rangle \in B_g \land \langle x, n_2 + 1 \rangle \notin B_g \}
\]
Theorem 5.6 ($3 \Rightarrow 2$)

- $B_f = \{ \langle x, n \rangle \mid f(x) \geq n \} \in \text{PP}$
- $B_g = \{ \langle x, n \rangle \mid g(x) \geq n \} \in \text{PP}$

Now consider:

$V = \{ \langle x, n_1, n_2 \rangle \mid f(x) \geq n_1 \land f(x) < n_1 + 1 \land \langle x, n_2 \rangle \in B_g \land \langle x, n_2 + 1 \rangle \notin B_g \}$

3. UP = PP

2. \#P is closed under every polynomial-time computable operation
Theorem 5.6 (3 ⇒ 2)

- $B_f = \{ \langle x, n \rangle | f(x) \geq n \} \in \text{PP}$
- $B_g = \{ \langle x, n \rangle | g(x) \geq n \} \in \text{PP}$

Now consider:

$V = \{ \langle x, n_1, n_2 \rangle | f(x) = n_1 \land \langle x, n_2 \rangle \in B_g \land \langle x, n_2 + 1 \rangle \notin B_g \}$

3. UP = PP

2. \#P is closed under every polynomial-time computable operation
Theorem 5.6 \((3 \Rightarrow 2)\)

- \(B_f = \{ \langle x, n \rangle \mid f(x) \geq n \} \in \text{PP}\)
- \(B_g = \{ \langle x, n \rangle \mid g(x) \geq n \} \in \text{PP}\)

Now consider:

\[ V = \{ \langle x, n_1, n_2 \rangle \mid \langle x, n_1 \rangle \in B_f \land \langle x, n_1 + 1 \rangle \notin B_f \land \langle x, n_2 \rangle \in B_g \land \langle x, n_2 + 1 \rangle \notin B_g \} \]

3. \(\text{UP} = \text{PP}\)

2. \#P is closed under every polynomial-time computable operation
3. UP = PP

2. \#P is closed under every polynomial-time computable operation

**Theorem 5.6** (3 \(\Rightarrow\) 2)

- \(B_f = \{\langle x, n \rangle \mid f(x) \geq n\} \in \text{PP}\)
- \(B_g = \{\langle x, n \rangle \mid g(x) \geq n\} \in \text{PP}\)

Now consider:

\[ V = \{\langle x, n_1, n_2 \rangle \mid \langle x, n_1 \rangle \in B_f \wedge \langle x, n_1 + 1 \rangle \notin B_f \wedge \langle x, n_2 \rangle \in B_g \wedge \langle x, n_2 + 1 \rangle \notin B_g \} \]
Theorem 5.6 (3 $\Rightarrow$ 2)

- $B_f = \{ \langle x, n \rangle \mid f(x) \geq n \} \in \text{PP}$
- $B_g = \{ \langle x, n \rangle \mid g(x) \geq n \} \in \text{PP}$

Now consider:

\[ V = \{ \langle x, n_1, n_2 \rangle \mid \langle x, n_1 \rangle \in B_f \land \langle x, n_1 + 1 \rangle \notin B_f \land \langle x, n_2 \rangle \in B_g \land \langle x, n_2 + 1 \rangle \notin B_g \} \]
Theorem 5.6 ($3 \Rightarrow 2$)

- $B_f = \{ \langle x, n \rangle | f(x) \geq n \} \in \text{PP}$
- $B_g = \{ \langle x, n \rangle | g(x) \geq n \} \in \text{PP}$

Now consider:

$$V = \{ \langle x, n_1, n_2 \rangle | \langle x, n_1 \rangle \in B_f \land \langle x, n_1 + 1 \rangle \notin B_f \land \langle x, n_2 \rangle \in B_g \land \langle x, n_2 + 1 \rangle \notin B_g \}$$

As $V$ 4-truth table reduces to $B_f \oplus B_g$, and as PP is closed under bounded-truth table reductions we have that $V \in \text{PP}$ (alternatively $\text{UP} = \text{PP} \Rightarrow \text{UP} = \text{P}^{\text{PP}}$)

As we are assuming $\text{UP} = \text{PP}$, we have that $V \in \text{UP}$

3. $\text{UP} = \text{PP}$

2. $\#P$ is closed under every polynomial-time computable operation
Theorem 5.6 ($3 \Rightarrow 2$)

\[ V = \{ \langle x, n_1, n_2 \rangle \mid \langle x, n_1 \rangle \in B_f \wedge \langle x, n_1 + 1 \rangle \notin B_f \wedge \langle x, n_2 \rangle \in B_g \wedge \langle x, n_2 + 1 \rangle \notin B_g \} \]

3. $UP = PP$

2. $\#P$ is closed under every polynomial-time computable operation
Theorem 5.6 (3 \Rightarrow 2)

V = \{ \langle x, n_1, n_2 \rangle \mid \langle x, n_1 \rangle \in B_f \land \langle x, n_1 + 1 \rangle \notin B_f \land \langle x, n_2 \rangle \in B_g \land \langle x, n_2 + 1 \rangle \notin B_g \} 

Recall that \( f \) and \( g \) are arbitrary \#P functions, so there is a polynomial \( q \) such that, for all \( x \),
\[
\max \{ f(x), g(x) \} \leq 2^{q(|x|)}
\]

3. \( \text{UP} = \text{PP} \)

2. \#P is closed under every polynomial-time computable operation
Theorem 5.6 ($3 \Rightarrow 2$)

\[ V = \{ \langle x, n_1, n_2 \rangle \mid \langle x, n_1 \rangle \in B_f \land \langle x, n_1 + 1 \rangle \notin B_f \land \langle x, n_2 \rangle \in B_g \land \langle x, n_2 + 1 \rangle \notin B_g \} \]

Recall that $f$ and $g$ are arbitrary $\#P$ functions, so there is a polynomial $q$ such that, for all $x$,$\text{max}\{f(x), g(x)\} \leq 2^{q(|x|)}$

Consider an NPTM machine $N$ which on input $x$ does the following:

3. $UP = PP$

2. $\#P$ is closed under every polynomial-time computable operation
Theorem 5.6 ($3 \Rightarrow 2$)

$$V = \{ \langle x, n_1, n_2 \rangle \mid \langle x, n_1 \rangle \in B_f \land \langle x, n_1 + 1 \rangle \notin B_f \land \langle x, n_2 \rangle \in B_g \land \langle x, n_2 + 1 \rangle \notin B_g \}$$

Recall that $f$ and $g$ are arbitrary $\#P$ functions, so there is a polynomial $q$ such that, for all $x$,

$$\max \{ f(x), g(x) \} \leq 2^{q(|x|)}$$

Consider an NPTM machine $N$ which on input $x$ does the following:

- Nondeterministically choose an integer $i$, $0 \leq i \leq 2^{q(|x|)}$
Theorem 5.6 (3 ⇒ 2)

\[ V = \{ \langle x, n_1, n_2 \rangle \mid \langle x, n_1 \rangle \in B_f \land \langle x, n_1 + 1 \rangle \notin B_f \land \langle x, n_2 \rangle \in B_g \land \langle x, n_2 + 1 \rangle \notin B_g \} \]

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Consider an NPTM machine $N$ which on input $x$ does the following:

- Nondeterministically choose an integer $i$, $0 \leq i \leq 2^{q(|x|)}$

3. $UP = PP$

2. $\#P$ is closed under every polynomial-time computable operation
Theorem 5.6 ($3 \Rightarrow 2$)

$$V = \{ \langle x, n_1, n_2 \rangle \mid \langle x, n_1 \rangle \in B_f \land \langle x, n_1 + 1 \rangle \notin B_f \land \langle x, n_2 \rangle \in B_g \land \langle x, n_2 + 1 \rangle \notin B_g \}$$

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- Nondeterministically choose an integer $j$, $0 \leq j \leq 2^{q(|x|)}$

3. $\text{UP} = \text{PP}$

2. $\#P$ is closed under every polynomial-time computable operation
Theorem 5.6 \((3 \Rightarrow 2)\)

\[ V = \{ \langle x, n_1, n_2 \rangle \mid \langle x, n_1 \rangle \in B_f \land \langle x, n_1 + 1 \rangle \not\in B_f \land \langle x, n_2 \rangle \in B_g \land \langle x, n_2 + 1 \rangle \not\in B_g \} \]

Recall that \(f\) and \(g\) are arbitrary \(#P\) functions, so there is a polynomial \(q\) such that, for all \(x\),

\[ \max \{ f(x), g(x) \} \leq 2^{q(|x|)} \]

Consider an NPTM machine \(N\) which on input \(x\) does the following:

- Nondeterministically choose an integer \(i, 0 \leq i \leq 2^{q(|x|)}\)
- Nondeterministically choose an integer \(j, 0 \leq j \leq 2^{q(|x|)}\)

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- Nondeterministically guess a computation path of the UP machine for $V$ on input $\langle x, i, j \rangle$. If the path rejects, then reject.
Theorem 5.6 (3 ⇒ 2)

\[ V = \{ \langle x, n_1, n_2 \rangle \mid \langle x, n_1 \rangle \in B_f \land \langle x, n_1 + 1 \rangle \notin B_f \land \langle x, n_2 \rangle \in B_g \land \langle x, n_2 + 1 \rangle \notin B_g \} \]

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- Nondeterministically choose an integer \( j, 0 \leq j \leq 2^{q(|x|)} \)
- Nondeterministically guess a computation path of the \( UP \) machine for \( V \) on input \( \langle x, i, j \rangle \). If the path rejects, then reject. Otherwise, nondeterministically guess an integer \( k, 1 \leq k \leq op(i, j) \), and accept
3. $\text{UP} = \text{PP}$

2. $\#P$ is closed under every polynomial-time computable operation

Theorem 5.6 ($3 \Rightarrow 2$)

$V = \{ \langle x, n_1, n_2 \rangle \mid \langle x, n_1 \rangle \in B_f \land \langle x, n_1 + 1 \rangle \notin B_f \land \langle x, n_2 \rangle \in B_g \land \langle x, n_2 + 1 \rangle \notin B_g \}$

Recall that $f$ and $g$ are arbitrary $\#P$ functions, so there is a polynomial $q$ such that, for all $x$, $\max\{f(x), g(x)\} \leq 2^{q(|x|)}$

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- Nondeterministically guess a computation path of the UP machine for $V$ on input $\langle x, i, j \rangle$. If the path rejects, then reject. Otherwise, nondeterministically guess an integer $k$, $1 \leq k \leq \text{op}(i, j)$, and accept
Theorem 5.6 ($3 \Rightarrow 2$)

\[ V = \{ \langle x, n_1, n_2 \rangle \mid \langle x, n_1 \rangle \in B_f \land \langle x, n_1 + 1 \rangle \notin B_f \land \langle x, n_2 \rangle \in B_g \land \langle x, n_2 + 1 \rangle \notin B_g \} \]

Recall that $f$ and $g$ are arbitrary $\#P$ functions, so there is a polynomial $q$ such that, for all $x$,

\[ \max\{f(x), g(x)\} \leq 2^q(|x|) \]

Consider an NPTM machine $N$ which on input $x$ does the following:

- Nondeterministically choose an integer $i$, $0 \leq i \leq 2^q(|x|)$
- Nondeterministically choose an integer $j$, $0 \leq j \leq 2^q(|x|)$
- Nondeterministically guess a computation path of the UP machine for $V$ on input $\langle x, i, j \rangle$. If the path rejects, then reject. Otherwise, nondeterministically guess an integer $k$, $1 \leq k \leq op(i, j)$, and accept

2. $\#P$ is closed under every polynomial-time computable operation

3. $UP = \text{PP}$
Theorem 5.6 (3 ⇒ 2)

\[ V = \{ \langle x, n_1, n_2 \rangle \mid \langle x, n_1 \rangle \in B_f \land \langle x, n_1 + 1 \rangle \not\in B_f \land \langle x, n_2 \rangle \in B_g \land \langle x, n_2 + 1 \rangle \not\in B_g \} \]

Recall that \( f \) and \( g \) are arbitrary \#P functions, polynomial \( q \) such that, for all \( x \), \( \max\{f(x), g(x)\} \leq 2^{q(|x|)} \)

Consider an NPTM machine \( N \) which on input \( x \) does the following:

- Nondeterministically choose an integer \( i, 0 \leq i \leq 2^{q(|x|)} \)
- Nondeterministically choose an integer \( j, 0 \leq j \leq 2^{q(|x|)} \)
- Nondeterministically guess a computation path of the UP machine for \( V \) on input \( \langle x, i, j \rangle \). If the path rejects, then reject. Otherwise, nondeterministically guess an integer \( k, 1 \leq k \leq \text{op}(i, j) \), and accept
Theorem 5.6 (3 ⇒ 2)

\[ V = \{ \langle x, n_1, n_2 \rangle \mid \langle x, n_1 \rangle \in B_f \land \langle x, n_1 + 1 \rangle \notin B_f \land \langle x, n_2 \rangle \in B_g \land \langle x, n_2 + 1 \rangle \notin B_g \} \]

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- Nondeterministically choose an integer \( i \), \( 0 \leq i \leq 2^{q(|x|)} \)
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- Nondeterministically guess a computation path of the UP machine for \( V \) on input \( \langle x, i, j \rangle \). If the path rejects, then reject. Otherwise, nondeterministically guess an integer \( k \), \( 1 \leq k \leq op(i, j) \), and accept

So, for all \( x \), \( N(x) \) is an NPTM with exactly \( h(x) = op(f(x), g(x)) \) accepting paths.
Theorem 5.6 (3 ⇒ 2)

V = {⟨x, n₁, n₂⟩ | ⟨x, n₁⟩ ∈ B_f ∧ ⟨x, n₁ + 1⟩ ∉ B_f ∧ ⟨x, n₂⟩ ∈ B_g ∧ ⟨x, n₂ + 1⟩ ∉ B_g}

Recall that f and g are arbitrary #P functions, polynomial q such that, for all x, max{f(x), g(x)} ≤ 2^{q(|x|)}

Consider an NPTM machine N which on input x does the following:

- Nondeterministically choose an integer i, 0 ≤ i ≤ 2^{q(|x|)}
- Nondeterministically choose an integer j, 0 ≤ j ≤ 2^{q(|x|)}
- Nondeterministically guess a computation path of the UP machine for V on input ⟨x, i, j⟩. If the path rejects, then reject. Otherwise, nondeterministically guess an integer k, 1 ≤ k ≤ op(i, j), and accept

So, for all x, N(x) is an NPTM with exactly h(x) = op(f(x), g(x)) accepting paths. (So?)

2. #P is closed under every polynomial-time computable operation

3. UP = PP
Theorem 5.6 (3 ⇒ 2)

\[ V = \{ \langle x, n_1, n_2 \rangle \mid \langle x, n_1 \rangle \in B_f \land \langle x, n_1 + 1 \rangle \notin B_f \land \langle x, n_2 \rangle \in B_g \land \langle x, n_2 + 1 \rangle \notin B_g \} \]

Recall that \( f \) and \( g \) are arbitrary \( \#P \) functions, polynomial \( q \) such that, for all \( x, \max\{f(x), g(x)\} \leq 2^{q(|x|)} \)

Consider an NPTM machine \( N \) which on input \( x \) does the following:

- Nondeterministically choose an integer \( i, 0 \leq i \leq 2^{q(|x|)} \)
- Nondeterministically choose an integer \( j, 0 \leq j \leq 2^{q(|x|)} \)
- Nondeterministically guess a computation path of the UP machine for \( V \) on input \( \langle x, i, j \rangle \). If the path rejects, then reject. Otherwise, nondeterministically guess an integer \( k, 1 \leq k \leq \text{op}(i, j) \), and accept

So, for all \( x, N(x) \) is an NPTM with exactly \( h(x) = \text{op}(f(x), g(x)) \) accepting paths. \( h(x) \) is a \( \#P \) function!
Theorem 5.6 (3 $\Rightarrow$ 2)

Let $op: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be an arbitrary polynomial-time computable operation

Let $f$ and $g$ be arbitrary $\#P$ functions

Our goal now is to show that $h(x) = op(f(x), g(x))$ is also a $\#P$ function

3. $UP = PP$

2. $\#P$ is closed under every polynomial-time computable operation
Theorem 5.6 (3 ⇒ 2)

Let $op : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be an arbitrary polynomial-time computable operation.

Let $f$ and $g$ be arbitrary $\#P$ functions.

Our goal now is to show that $h(x) = op(f(x), g(x))$ is also a $\#P$ function.

So, as $op$ was an arbitrary polynomial-time computable operation, we have shown that $\#P$ is closed under every polynomial-time operation.
The following statements are equivalent:

1. \( \#P \) is closed under proper subtraction.
2. \( \#P \) is closed under every polynomial-time computable operation
3. \( UP = PP \)
So what does \( \text{UP} = \text{PP} \) mean?
First, a new class: $\oplus P$

$\oplus P = \{ L | (\exists \text{NPTM} N (\forall x)[x \in L \Leftrightarrow \#acc_N(x) \not\equiv 0 \text{ (mod 2)}] \}$

So:

- $x \in L \Leftrightarrow \#acc_N(x)$ is odd
- $x \notin L \Leftrightarrow \#acc_N(x)$ is even
Hierarchy

Theorem 5.7:

1. $UP = PP$

2. $UP = NP = \text{coNP} = PH = \varpi P = PP \cup PP^{PP} \cup PP^{PPP} \cup \ldots$
Hierarchy

Theorem 5.7 (Proof):

\[
UP \subseteq NP \subseteq PP
\]

PP is closed under complementation, \( coNP \subseteq PP \)

\[
UP \subseteq coNP \subseteq PP
\]

What happens if \( UP = PP \)?
Hierachy

Theorem 5.7 (Proof):

We already know if $\text{NP} = \text{coNP}$, then $\text{PH} = \text{NP}$.

If $\text{UP} = \text{PP}$, then $\text{UP} = \text{NP} = \text{PP} = \text{coNP} = \text{PH}$
Hierarchy

Theorem 5.7 (Proof):

Considering $P_{UP}$

$P_{UP} \subseteq PH$

(With $PH = UP$)

$P_{UP} = UP$
Lemma 4.14

\[ \text{PP} \oplus \text{P} \subseteq \text{P}^{\#P[1]}, \text{and } \text{BPP} \oplus \text{P} \subseteq \text{P}^{\#P[1]} \]

Proof continuing

\[ \text{P}^{\#P[1]} \subseteq \text{P}^{\#P} = \text{PP} \]

\[ \text{PP} \oplus \text{P} \subseteq \text{PP} \quad \text{PP} \oplus \text{P} \subseteq \text{P}^{\text{UP}} \text{ (with PP = UP)} \]

\[ \text{PP} \oplus \text{P} \subseteq \text{UP} \text{ (with P}^{\text{UP}} \text{ = UP)} \]
Hierarchy

Theorem 5.7 (Proof):

\[ PP \oplus P = UP \]

\[ PP \oplus P = PP = \oplus P = UP \] (With \( \oplus P \) is ‘between’ PP and UP)
Theorem 5.7 (Proof):

Recall a conclusion from Lemma 4.14: $PP^{\oplus P} \subseteq PP$

$PP^{PP} \subseteq PP^{PP} \subseteq PP^{\oplus P}$

$PP^{PP^{PP}} \subseteq PP^{PP^{\oplus P}} \subseteq PP^{PP} \subseteq PP^{PP} \subseteq PP^{\oplus P} = UP$
One more to go:

Integer Division
Integer Division

The arithmetic integer division is an operation from $\odot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, defined as $a \odot b = \lfloor a/b \rfloor$. 
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- An NPTM only has integer number of accepting paths, so let’s not worry about fractional division;
Integer Division

The arithmetic integer division is an operation from $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$, defined as $a \odot b = \lfloor a/b \rfloor$.

- An NPTM only has integer number of accepting paths, so let’s not worry about fractional division;
- To avoid division by zero, we will adopt a revised definition of closure for NPTM (Definition 5.8):

Let $\mathcal{F} : \mathbb{N} \rightarrow \mathbb{N}$ be a class of functions. We say that $\mathcal{F}$ is closed under integer division ($\odot$) if

$$(\forall f_1 \in \mathcal{F})(\forall f_2 \in \mathcal{F} : (\forall n)[f_2(n) > 0])[f_1 \odot f_2 \in \mathcal{F}].$$
The following statements are equivalent:

1. \#P is closed under **integer division**;
2. \#P is closed under every polynomial-time computable operation;
3. \( \text{UP} = \text{PP} \).
Theorem 5.9

The following statements are equivalent:

1. \( \#P \) is closed under integer division;
2. \( \#P \) is closed under every polynomial-time computable operation;
3. \( \text{UP} = \text{PP} \).

We know 2 implies 1, as integer division is a polynomial-time computable operation; and we have proved 3 implies 2. So we only need to prove 1 implies 3, “\( \#P \) is closed under integer division \( \rightarrow \) \( \text{UP} = \text{PP} \)”.
Theorem 5.9 proof

Suppose \#P is closed under integer division; let \( L \in \text{PP} \). Then, there exists an NPTM \( N \) and an integer \( k \geq 1 \) such that ——

1. On each input \( x \), \( N(x) \) has exactly \( 2^{|x|^k} \) computation paths, each containing exactly \( |x|^k \) binary choices;
Theorem 5.9 proof

Suppose \#P is closed under integer division; let \( L \in \text{PP} \). Then, there exists an NPTM \( N \) and an integer \( k \geq 1 \) such that ——

1. On each input \( x \), \( N(x) \) has exactly \( 2^{|x|^k} \) computation paths, each containing exactly \( |x|^k \) binary choices;
2. On each input \( x, x \in L \) if and only if \( N(x) \) has at least \( 2^{|x|^{k-1}} \) accepting paths;
Theorem 5.9 proof

Suppose \( \#\text{P} \) is closed under integer division; let \( L \in \text{PP} \). Then, there exists an NPTM \( N \) and an integer \( k \geq 1 \) such that ——

1. On each input \( x \), \( N(x) \) has exactly \( 2^{|x|^k} \) computation paths, each containing exactly \( |x|^k \) binary choices;
2. On each input \( x, x \in L \) if and only if \( N(x) \) has at least \( 2^{|x|^k-1} \) accepting paths;
3. On each input \( x, N(x) \) has at least one rejecting path.
Consider three auxiliary functions (by construction):

\[ f, g, h, \]

- \( f(x) \) calculates the number of accepting paths of \( N(x) \);
- \( g(x) = 2^{|x|^k-1} \);
- \( h(x) = f(x) \ominus g(x) \).
Theorem 5.9 proof

(cont.) Consider three auxiliary functions (by construction):

\[ f, g, h, \]

- \(f(x)\) calculates the number of accepting paths of \(N(x)\);
- \(g(x) = 2^{|x|^k - 1}\);
- \(h(x) = f(x) \otimes g(x)\).

\(f(x), g(x)\) are indeed \#P functions.
Theorem 5.9 proof

(cont.) Consider three auxiliary functions (by construction):

\[ f, g, h, \]

- \( f(x) \) calculates the number of accepting paths of \( N(x) \);
- \( g(x) = 2^{|x|^k-1} \);
- \( h(x) = f(x) \otimes g(x) \).

\( f(x), g(x) \) are indeed \#P functions.

By assumption of closure, so is \( h(x) \).
Theorem 5.9 proof

(cont.) Note that,

If \( x \in L \):

\( N(x) \) has at least \( 2^{|x|^k - 1} \) accepting paths, then \( 2^{|x|^k} > f(x) \geq 2^{|x|^k - 1} \).

Then, \( h(x) = f(x) \otimes g(x) = \lfloor f(x)/g(x) \rfloor = 1 \).
Theorem 5.9 proof

(cont.) Note that,

If \( x \in L \):

\[
N(x) \text{ has at least } 2^{|x|^k} - 1 \text{ accepting paths, then } 2^{|x|^k} > f(x) \geq 2^{|x|^k} - 1.
\]

Then, \( h(x) = f(x) \bigcirc g(x) = \lfloor f(x)/g(x) \rfloor = 1 \).

If \( x \notin L \):

\[
N(x) \text{ has less than } 2^{|x|^k} - 1 \text{ accepting paths, then } 2^{|x|^k} - 1 > f(x).
\]

Then, \( h(x) = f(x) \bigcirc g(x) = \lfloor f(x)/g(x) \rfloor = 0 \).
Theorem 5.9 proof

(cont.) In summary, we found function $h$ behaves as the following:

On input $x$,

$$h(x) =$$

1, if $x \in L$;

0, if $x \notin L$. 
Theorem 5.9 proof

(cont.) In summary, we found function $h$ behaves as the following:

On input $x$,

$$h(x) = \begin{cases} 
1, & \text{if } x \in L; \\
0, & \text{if } x \notin L.
\end{cases}$$

By closure property, we successfully constructed a “decider” of $L$ that’s “unambiguous”!
Theorem 5.9 proof

(cont.) The nondeterministic machine corresponding to \( h \) is a "UP machine".

Thus, \( L \in \text{UP}; \#\text{P is closed under integer division} \rightarrow \text{UP} = \text{PP}. \) Thus, theorem 5.9 is proved.

Q.E.D.