CSC 246/446 Homework 1

January 15, 2015

Due in class Wed 1/21.

1. Markov's inequality is a handy tool for setting bounds on the probability that a random variable gets too big. It states that, for any random variable X having values that are always greater than or equal to zero, and for any $\delta \geq 0$,

$$P(X \geq \delta) \leq \frac{1}{\delta} E[X]$$

that is, the probability that X is greater than or equal to δ is less than or equal to the expectation of X divided by δ. Prove this inequality.

2. Use Markov's inequality to prove Chebyshev's inequality:

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

3. Suppose that the return r_i for a stock on day i is i.i.d. with each r_i distributed as follows:

$$r_i = \begin{cases} 1/2 & \text{w/ prob} 1/2 \\ 2 & \text{w/ prob} 1/2 \end{cases}$$

and the return of one dollar after N days is $R = \prod_{i=1}^{N} r_i$.

   (a) What is $\mathrm{E}[R]$?

   (b) Let $x_i = \log r_i$ and $X = \log R$. What is $\mathrm{E}[X]$?

   (c) What is $\mathrm{Var}[x_i]$?

   (d) What is $\mathrm{Var}[X]$?

   (e) Use Chebyshev's inequality to derive a bound on the probability that $X > (9/8)^N$. What is the limit of this probability as $N \to \infty$?

4. Now suppose that you invest in two stocks, each of which behaves independently of the other, doubling of halving each day as in the previous question. At the beginning of each day, you invest half your money in each stock.

   (a) What is the distribution over r_i, your return for day i?

   (b) As before, $R = \prod_{i=1}^{N} r_i$, $x_i = \log r_i$, and $X = \log R$. What is $\mathrm{E}[R]$?

   (c) What is $\mathrm{E}[X]$?

   (d) What is $\mathrm{Var}[x_i]$?

   (e) What is $\mathrm{Var}[X]$?

   (f) Use Chebyshev's inequality to derive a bound on the probability that $X < (9/8)^N$ (note direction of the inequality, as opposed to the previous question). What is the limit of this probability as $N \to \infty$?

   (g) Comparing your results for the two investment strategies, how are they similar and how are they different? Which of the two would you choose? Why?