Half \( (L) \) = \( \{ x \mid (\exists y) [lxl=lyl AND xy \in L] \} \).

Prove that if \( L \) is regular, then Half \( (L) \) is regular.

Do so by showing how, given any DFA (as a 5-tuple) to build in terms of it an NFA that accepts Half \( (L) \), where \( L \) is the language decided (i.e., accepted) by the DFA.
Construction: \( \hat{\Sigma} = \Sigma \). \( \hat{Q} = \{S\} \cup \{ (a, b) \mid a \in Q \text{ and } b \notin Q \} \).
\( \hat{q}_0 = S \). \( \hat{F} = \{ (a, b) \mid a = b \land a \in Q \land b \notin Q \} \).

Definition of \( \hat{\delta} \):
\( \hat{\delta}(S, \varepsilon) = \{ (q_0, c) \mid c \in F \} \).

For each \( d \in \Sigma \):
\( \hat{\delta}(a, b, d) = \{ (\delta(a, d), q) \mid (\exists e \in \Sigma) \hat{\delta}(q, e) = (a, b) \} \).

\( \hat{\delta}(a, b, \varepsilon) = \emptyset \)

For each \( d \in \Sigma \): \( \hat{\delta}(S, d, d) = \emptyset \).
\[ \hat{\xi}(a, b, d) = \exists (\mathcal{G}(a, d), q) \mid \\
(\exists e \in \mathcal{G}(\exists f \in \mathcal{G})) \\
\mathcal{G}(\mathcal{G}(q, e), f) = b \]
Prove that there is a regular set $L$ such that $\text{Bookends}(L)$ is not regular.

Hint: Use a closure property of the regular sets, and the pumping lemma.
One solution:
\[ L = 0^*100^*10^* \text{ is regular.} \]

Note that \( \text{Bookends} (L) \cap 0^*110^* = \{0^k110^k \mid k \geq 3\}. \] (8)

101 yields 11. 010010 yields 0110. 00100010 yields 001100. Etc.

And the fact that the 1s are adjacent means the only strings in the intersection on line (8) are ones where what was deleted was exactly the "00*" between the 1s, which for things in the intersection must have been exactly \( \frac{1}{3} \) the length. So the RHS is all the strings in the intersection.

But \( 0^*110^* \) is regular! So if \( \text{Bookends} (L) \) is regular, then since the regular sets are closed under intersection, \( L' = \{0^k110^k \mid k \geq 3\} \) is regular. But \( L' \) is easily proven nonregular by a routine use of the Pumping Lemma. So we conclude that \( \text{Bookends}(L) \) is not regular, despite the fact that \( L \) is regular!!!