Theorem: $L = \{ \langle M_1, M_2 \rangle \mid |L(M_1) \cap L(M_2)| \geq 2\} \cup \{ \langle M \rangle \mid |L(M)| = 2\}$ is undecidable.

Proof: Assume $L$ is decidable by decision $R$. We will use this assumption to build a TM $S$ that decides $\overline{A_{TM}}$.

$S$ is: a TM that on input $\langle M, x \rangle$ does the following:

1. Build TM $M_1$ with $L(M_1) = \Sigma^*$.
2. Build a TM $M_2$ that does the following:
   - On a random input $y$, if $y$ is one of the 280 lexically smallest strings in $\Sigma^*$, then the TM successively simulates $M_{\langle M \rangle}$.
   - Else reject.
3. Run $R(\langle M_1, M_2 \rangle)$ if it accepts then accept, else reject.

End of construction of $S$. $S$ is a reducer for $\overline{A_{TM}}$. Contradiction!
Let $L' = \{ \langle M_1, M_2 \rangle \mid |L(M_1) \cap L(M_2)| \geq 280 \}$ be Turing-recognizable.

**Problem:** Let's build a TM $M$, such that $L(M) = L'$. That suffices.

Then $M$ will be the TM that on input $\langle M_1, M_2 \rangle$ does the following:

1. Build an enumerating TM, $E_1$, for $L(M_1)$.
2. Build an enumerating TM, $E_2$, for $L(M_2)$.
3. Run $E_1$ for 1 step, then $E_2$ for 2 steps, then $E_1$ for 3 steps, then $E_2$ for 4 steps, then $E_1$ for 5 steps, ....
4. And, as we do this, we collect separately the things $E_1$ puts into $L(M_1)$ and that $E_2$ puts into $L(M_2)$.
5. And the first moment when the intersection of these 2 collections (considered as sets) has $\geq 280$ elements, accept. End of proof.

**Specification:** $L(M) = L'$ (verbal argument was just given.)
Def \( L \) is co-Turing-recognizable (aka \( L \) is coRE) exactly if \( \overline{L} \) is Turing-recognizable.

Example: We know \( A_{TM} \) is Turing-recognizable. Thus \( \overline{A_{TM}} \) is co-Turing-recognizable.

\( L'' = \{ \langle M \rangle \mid M \) is a TM and \( L(M) = \emptyset \} \) is co-Turing-recognizable.

Proof: We will show that \( L'' \) is Turing-recognizable. Given input \( \langle M' \rangle \), if \( M' \) is not a valid TM accept. Otherwise, build the TM \( M \) for \( L(M) \), call \( \text{HE} \) simulate \( E \). If the moment it enumerates any string \( \langle M' \rangle \), halt. \( L'' \) is Turing-recognizable.

So clearly \( L'' \) is Turing-recognizable.