

# SEMANTICS OF FOL

Consider the inference

Dog(Snoopy),  $(\forall x \text{ Dog}(x) \Rightarrow \text{Has-Tail}(x))$   
Has-Tail(Snoopy)

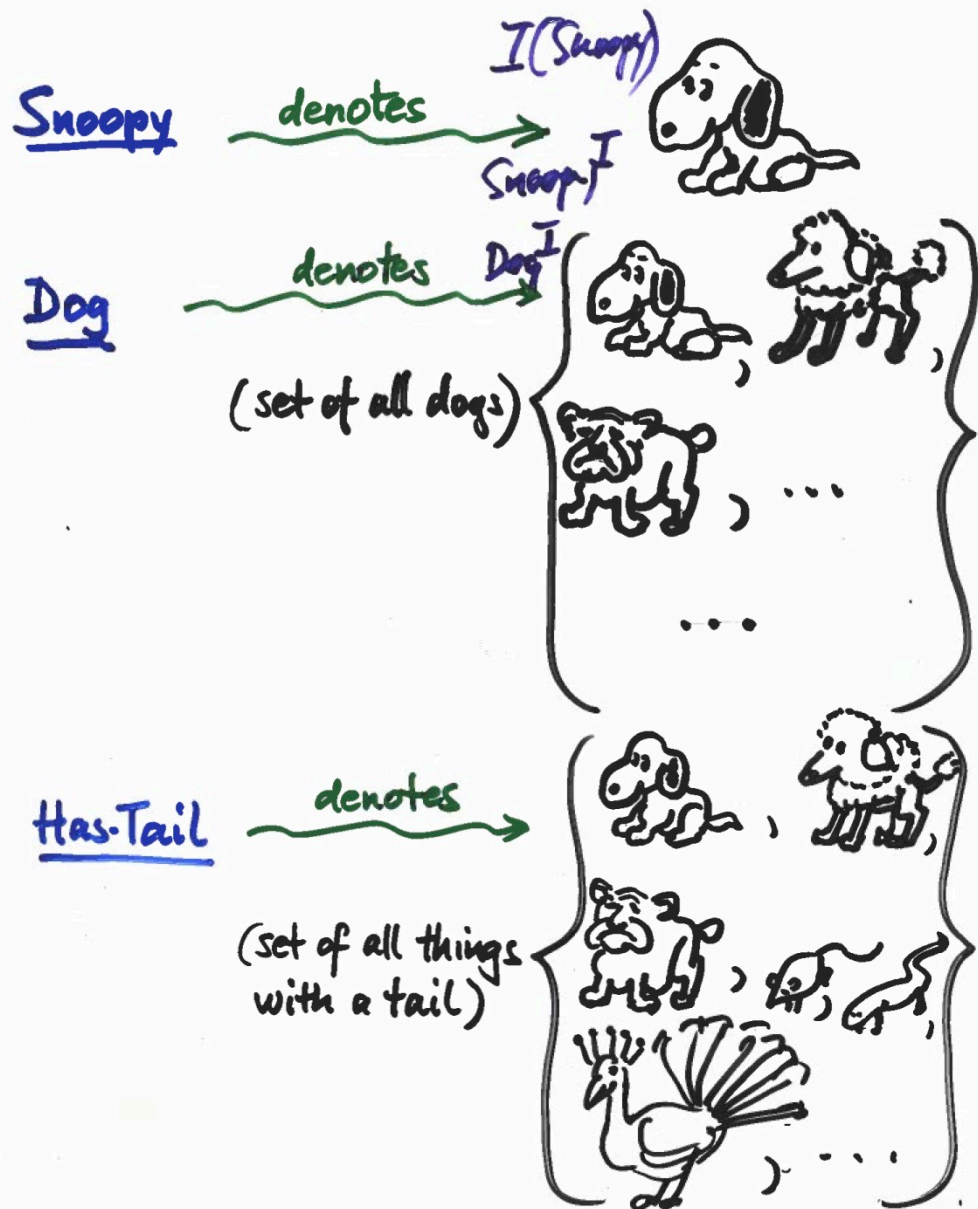
It seems that if the premises are true,  
so is the conclusion. Why? More generally,  
how can we distinguish such "truth-preserving"  
inferences from others??

We need a theory of truth for FOL sentences!

This only makes sense if FOL sentences are  
taken to be assertions about some domain.

This in turn requires that terms like  
Dog, Snoopy, Has-Tail, ... refer to (denote)  
something in the domain.

Here's how we can do it (example):



The "denotes" function, say  $I$ , is called an interpretation.

We say  $\text{Dog}(\text{Snoopy})$  is true under interpretation  $I$  iff

$$I(\text{Snoopy}) \in I(\text{Dog})$$

Clearly this is the case for  $\text{Dog}(\text{Snoopy})$ . We write (as a first approximation)

$$\vDash_I \text{Dog}(\text{Snoopy}).$$

For the given  $I$ , it's also clear that

$$\vDash_I \text{Has-Tail}(\text{Snoopy})$$

It's also the case (less obviously...) that

$$\vDash_I (\forall x (\text{Dog}(x) \Rightarrow \text{Has-Tail}(x)))$$

(Truth conditions for " $\forall$ " will be given later)

Another interpretation,  $I'$ :

Snoopy  $\xrightarrow{I'}$  4

Dog  $\xrightarrow{I'}$  {4, 8, 12, 16, ...}

Has-Tail  $\xrightarrow{I'}$  {1, 4, 6, 8, 9, 10, 12, 14, 15, ...}

All 3 formulas are again true under  $I'$ !

Formal "soundness" (truth-preservation) claim about the original inference:

For any interpretation  $I$  of Snoopy, Dog, Has-Tail (where  $I$  interprets Snoopy as an individual and Dog, Has-Tail as sets of individuals), if

$\models_I \text{Dog}(\text{Snoopy})$  and  $\models_I (\forall x \text{Dog}(x) \Rightarrow \text{Has-Tail}(x))$

then

$\models_I \text{Has-Tail}(\text{Snoopy})$

Let's get a little more precise & general.

First, before we talk about interpretations of particular symbols we should fix the domain  $\mathcal{D}$  of all the things we may wish to talk about in our FO language.

Then we interpret individual constants  $\tau$  as individuals in  $\mathcal{D}$ , i.e.,

$I(\tau) \in \mathcal{D}$  also written  $\tau^I \in \mathcal{D}$

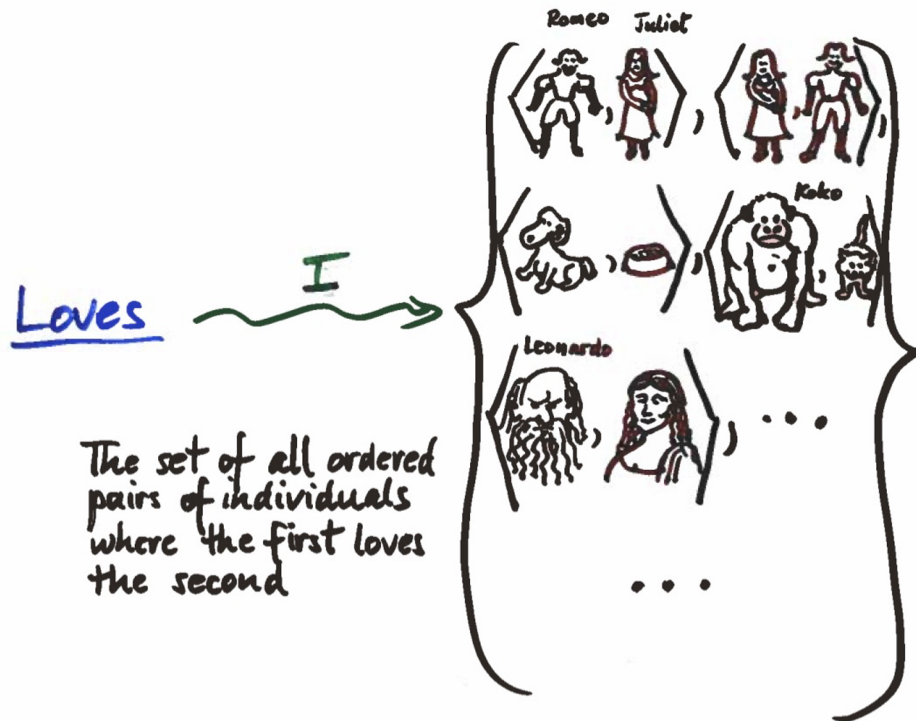
We interpret monadic (1-place) predicate constants  $\pi$  as sets of individuals in  $\mathcal{D}$ , i.e.,

$I(\pi) \subseteq \mathcal{D}$  also written  $\pi^I \subseteq \mathcal{D}$

For truth, we will now write

$\models_M \varphi$ ,  
where  $M = (\mathcal{D}, I)$ , the model.

What is  $I(\pi)$  for 2-place, 3-place, etc. predicates?



$$\begin{aligned}
 I(\text{Loves}) &\subseteq \mathcal{D} \times \mathcal{D} = \mathcal{D}^2 \\
 &\quad \uparrow \\
 &\text{or Loves}^I &= \{ \langle d_1, d_2 \rangle \mid d_1, d_2 \in \mathcal{D} \}
 \end{aligned}$$

$$\begin{aligned}
 \vDash_{\mathcal{M}} \text{Loves}(\text{Romeo}, \text{Juliet}) &\quad \text{iff} \\
 \langle \text{Romeo}^I, \text{Juliet}^I \rangle &\in \text{Loves}^I
 \end{aligned}$$

First-order logic vs. a first-order language  
 fixed vocabulary of constants

