

CSC2/455 Software Analysis and Improvement

An Introduction to SAT/SMT Solvers

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April 10, 2023

URCS

Outline

Introduction

SAT Solving

SMT Solvers

Applications to Program Analysis

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SAT Solving

SMT Solvers

Applications to Program Analysis

So far

- So far:
 - Iterative Data Flow Analysis
 - Type Analysis
 - Region Analysis
 - Abstract Interpretation
- Today
 - Satisfiability (SAT) Solvers
 - Satisfiability Modulo Theories (SMT) Solvers
- Next:
 - Model Checking
 - Symbolic Execution
 - Hoare Logic

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The Satisfiability (SAT) Problem

Given a formula in propositional logic (variables, true, false, \wedge , \vee , \neg , and parentheses), is there an assignment to variables that makes the formula true?

- $(A \vee B \vee C) \wedge (\neg A \vee B)$ (conjunctive normal form, CNF)
- $(A \wedge B \wedge C) \vee (\neg A \wedge B)$ (disjunctive normal form, DNF)
- $A \wedge \neg A$ (CNF)

Solutions

- $(A \vee B \vee C) \wedge (\neg A \vee B)$
 - $B = \text{true}$ is required in any satisfying assignment (A and C don't matter)
- $(A \wedge B \wedge C) \vee (\neg A \wedge B)$
 - A, B, C all true is one satisfying assignment
 - $A = \text{false}$ and $B = \text{true}$ is another satisfying assignment
- $A \wedge \neg A$ is obviously unsatisfiable

3-SAT is NP-Complete

- If the maximum number of variables in a clause of a CNF formula is k , we call that problem k -SAT
- 2-SAT is solvable in polynomial time
- 3-SAT is NP-complete
 - In worst case, must explore every possible assignment of values to each variable

SAT Problem Sizes

- There are many good SAT solvers now available
 - Based on the Davis–Putnam-Logemann-Loveland (DPLL) algorithm
 - Often enhanced with Conflict-Driven Clause Learning (CDCL)
 - SAT is decidable, if untractable
- Intractability not a hindrance usually
 - Can scale to very large problems
 - Millions of clauses
 - See: The International SAT Competition
- Applied to many hardware and software verification problems
- SAT solvers return:
 - SAT: if a satisfying assignment is found (and the values that satisfy the proposition)
 - UNSAT: if no satisfying assignment exists

Proving statements involving Propositional Logic

Prove $\neg(A \wedge B) = (\neg A \vee \neg B)$

A	B	$A \wedge B$	$P = \neg(A \wedge B)$	$\neg A$	$\neg B$	$Q = \neg A \vee \neg B$	$P \iff Q$
F	F	F	T	T	T	T	T
F	T	F	T	T	F	T	T
T	F	F	T	F	T	T	T
T	T	T	F	F	F	F	T

The statement $\neg(A \wedge B) = (\neg A \vee \neg B)$ is valid, it is true for all values of A and B .

Proof using a SAT solver

- $\neg(A \wedge B) \implies (\neg A \vee \neg B)$
- $(\neg A \vee \neg B) \implies \neg(A \wedge B)$
- Recall that $P \implies Q$ can be written as $\neg P \vee Q$
 - $\neg\neg(A \wedge B) \vee (\neg A \vee \neg B)$
 - $\neg(\neg A \vee \neg B) \vee \neg(A \wedge B)$
- So we have:
 - $R = (A \wedge B) \vee (\neg A \vee \neg B)$
 - $S = \neg(\neg A \vee \neg B) \vee \neg(A \wedge B)$
- For the proof, we need both R and S to be valid
 - If R is valid, what can we say about the satisfiability of $\neg R$?

The Satisfiability of $\neg R$

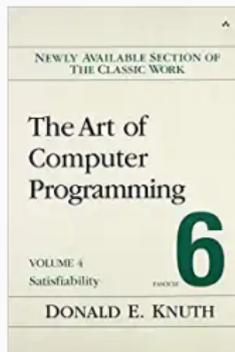
A	B	$P = A \wedge B$	$\neg A$	$\neg B$	$Q = \neg A \vee \neg B$	$P \vee Q$	$\neg(P \vee Q)$
F	F	F	T	T	T	T	F
F	T	F	T	F	T	T	F
T	F	F	F	T	T	T	F
T	T	T	F	F	F	T	F

If R is valid, then $\neg R$ is unsatisfiable!

- To prove a statement using a SAT solver:
 - Convert the statement to propositional logic
 - Negate it, and check for unsatisfiability
- Interesting corollary:
 - If the formula is SAT (implying the statement is false), the values that satisfy the statement are counter-examples

More details on SAT Solving (Book)

- Volume 4 Facsimile 6 of *The Art of Computer Programming*



More details on SAT/SMT Solving (Papers)

- Weissenbacher, Subramanyan, and Malik, Boolean Satisfiability Solvers: Techniques and Extensions
- Barrett, Sebastiani, Sheshia and Tinelli, Satisfiability Modulo Theories

Quantifiers, Theories

- Propositional logic can be extended with quantifiers
 - \exists , existential quantifier
 - \forall , universal quantifier
 - This is First-order Logic (FOL)
 - FOL is undecidable in general
- Both propositional logic and first-order logic are still boolean
- Can be extended with theories:
 - Arithmetic: adds numbers, operators $+$, $-$, \times , associativity, commutativity, etc.
 - Functions: adds $f(x)$
 - Bitvectors: model variables containing n bits (where $n > 1$)

Satisfiability Modulo Theories (SMT)

- A SMT solver checks for satisfiability *in a theory*
 - Can think of statements as propositional logic + theory
- Allows construction of “richer” statements
 - Can formulate propositions over integers, reals, etc.
 - Can use operators like $+$, $-$, \times , etc.
- Example: $\forall_{x,y} x > y \implies x + 1 > y + 1$
 - True over integers (\mathbb{Z})
 - False over machine integers/bitvectors

Decidability

- Some theories are decidable
 - Presburger arithmetic
- Most theories are undecidable
- However, some theories undecidable in general *are* decidable over quantifier-free fragments
- So, results of a SMT solver can be:
 - SAT
 - UNSAT
 - Don't know (or infinite loop)

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Some SMT solvers

- Microsoft Z3
 - Used to be available online at [rise4fun/Z3](http://rise4fun.com/Z3)
 - Available in most Linux distributions
- CVC5
- Yices
- Many more...

The SMT-LIB language

- Common input/output language for most SMT solvers
 - Some solvers support their own language as well
- Lisp-like
- Documented at smtlib.org
- Allows easy switching between solvers
 - We will use Z3 for the most part

Encoding a problem for Z3 to solve

Let's encode $\neg R$ from the previous example:

```
(declare-fun B () Bool)
(declare-fun A () Bool)
(assert (not (or (and A B) (not A) (not B))))
(check-sat)
```

And we run it:

```
$ z3 p1.smt
unsat
```

Alternative: Python library

```
#!/usr/bin/env python3

from z3 import *

s = Solver()
A, B = Booleans('A B')

R = Or(And(A, B), Or(Not(A), Not(B)))
notR = Not(R)

s.add(notR)
print(s.check())

print(s.sexpr()) # prints the SMTLIB code
```

See: Bjørner et al., Programming Z3, for a nice introduction to programming Z3 using Python.

SAT/SMT as constraint satisfaction

- The Constraint Satisfaction Problem seeks to find an assignment of values to variables subject to constraints
 - Each variable has a domain of values
 - Pick a variable, assign it a value, subject to constraints
 - If all variables can be assigned values, SAT else backtrack
- For SAT in propositional logic:
 - Two values, True and False
 - Constraint: formula must evaluate to true

Other Problems

- Dennis Yurichev's free book "SAT/SMT by Example" is a wonderful collection of examples
 - Minesweeper
 - Sudoku
 - Test case generation, etc.

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SMT Solvers in Program Analysis

- Express program behaviour in some logic
- Express program property in that logic
- Check if the property holds (i.e. is valid)

Assertions

```
int min(int a, int b) {
    if(a < b)
        return a;
    else
        return b;
}

int x, y, r;

r = min(x, y);

assert(r == x || r == y);
assert(r <= x && r <= y);
```

- These assertions test that `min` always returns the minimum of `x` and `y`
- But `assert` executes at runtime
- We will seek to prove statically:
 - $\forall_{x,y} (\min(x,y) = x \vee \min(x,y) = y) \wedge (\min(x,y) \leq x \wedge \min(x,y) \leq y)$
 - over all program paths

Checking the correctness of `min`

```
from z3 import *

s = Solver()
a, b, ret = Ints('a b ret')
ret = If(a < b, a, a)
#ret = If(a < b, a, b) # correct
cond = And(Or(ret == a, ret == b), And(ret <= a, ret <= b))
s.add(Not(cond))
print(s.sexpr())

if s.check() == sat:
    print("Incorrect. Counterexample: ", s.model())
else:
    print("Correct")
```

Output:

```
...
Incorrect. Counterexample: [b = 0, a = 1]
```

Proving Programs Equivalent

- If A is a source program and B is the compiled version, we would like to prove that $A = B$
 - This is called *translation validation*
 - What I've been doing for your submissions
- Undecidable, in general
- Not interesting only to compiler writers
 - If you take a piece of code and refactor it, did you break anything?

Test Case Generation

```
int min(int a, int b) {  
    if(a < b)  
        return a;  
    else  
        return a;  
}
```

```
int x, y, r;
```

```
r = min(1, 3);
```

```
...
```

- The test case (1, 3) is not sufficient to exercise all paths in the program
 - And it misses the bug!
- Can we find test cases to exercise all paths in the program?

Postscript

- SMT solvers are a marvellous piece of technology
- Learn to use one!