CSC2/455 Software Analysis and Improvement
An Introduction to SAT/SMT Solvers

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URCS

Outline

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So far

- So far:
  - Iterative Data Flow Analysis
  - Type Analysis
  - Region Analysis
  - Abstract Interpretation
- Today
  - Satisfiability (SAT) Solvers
  - Satisfiability Modulo Theories (SMT) Solvers
- Next:
  - Model Checking
  - Symbolic Execution
  - Hoare Logic

Outline

The Satisfiability (SAT) Problem

Given a formula in propositional logic (variables, true, false, $\wedge$, $\vee$, $\neg$, and parentheses), is there an assignment to variables that makes the formula true?

- $(A \vee B \vee C) \wedge (\neg A \vee B)$ (conjunctive normal form, CNF)
- $(A \wedge B \wedge C) \vee (\neg A \wedge B)$ (disjunctive normal form, DNF)
- $A \wedge \neg A$ (CNF)

Solutions

- $(A \vee B \vee C) \wedge (\neg A \vee B)$
  - $B = \texttt{true}$ is required in any satisfying assignment (A and C don't matter)
- $(A \wedge B \wedge C) \vee (\neg A \wedge B)$
  - A, B, C all true is one satisfying assignment
  - $A = \texttt{false}$ and $B = \texttt{true}$ is another satisfying assignment
- $A \wedge \neg A$ is obviously unsatisfiable

3-SAT is NP-Complete

- If the maximum number of variables in a clause of a CNF formula is k, we call that problem k-SAT
- 2-SAT is solvable in polynomial time
- 3-SAT is NP-complete
    - In worst case, must explore every possible assignment of values to each variable

SAT Problem Sizes

- There are many good SAT solvers now available
  - Based on the Davis–Putnam-Logemann-Loveland (DPLL) algorithm
  - Often enhanced with Conflict-Driven Clause Learning (CDCL)
  - SAT is decidable, if untractable
- Intractability not a hindrance usually
  - Can scale to very large problems
  - Millions of clauses
  - See: The International SAT Competition
- Applied to many hardware and software verification problems
- SAT solvers return:
  - SAT: if a satisfying assignment is found (and the values that satisfy the proposition)
  - UNSAT: if no satisfying assignment exists

Proving statements involving Propositional Logic

Prove $\neg(A \wedge B) = (\neg A \vee \neg B)$

A	B	$A \wedge B$	P=$\neg(A \wedge B)$	$\neg A$	$\neg B$	Q=$\neg A \vee \neg B$	$P \iff Q$
F	F	F	**T**	T	T	**T**	T
F	T	F	**T**	T	F	**T**	T
T	F	F	**T**	F	T	**T**	T
T	T	T	**F**	F	F	**F**	T

The statement $\neg(A \wedge B) = (\neg A \vee \neg B)$ is valid, it is true for all values of A and B.

Proof using a SAT solver

- $\neg(A \wedge B) \implies (\neg A \vee \neg B)$
- $(\neg A \vee \neg B) \implies \neg(A \wedge B)$
- Recall that $P \implies Q$ can be written as $\neg P \vee Q$
  - $\neg\neg(A \wedge B) \vee (\neg A \vee \neg B)$
  - $\neg(\neg A \vee \neg B) \vee \neg(A \wedge B)$
- So we have:
  - $R = (A \wedge B) \vee (\neg A \vee \neg B)$
  - $S = \neg(\neg A \vee \neg B) \vee \neg(A \wedge B)$
- For the proof, we need both R and S to be valid
  - If R is valid, what can we say about the satisfiability of $\neg R$?

The Satisfiability of $\neg R$

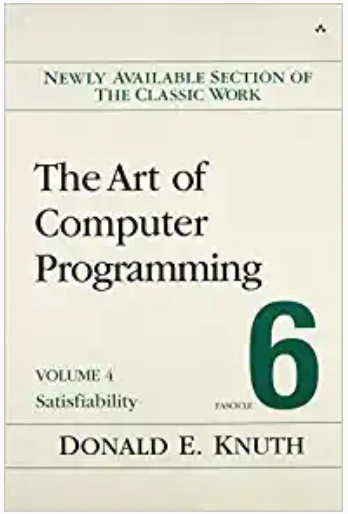
A	B	$P = A \wedge B$	$\neg A$	$\neg B$	$Q = \neg A \vee \neg B$	$P \vee Q$	$\neg(P \vee Q)$
F	F	F	T	T	T	T	F
F	T	F	T	F	T	T	F
T	F	F	F	T	T	T	F
T	T	T	F	F	F	T	F

If R is valid, then $\neg R$ is unsatisfiable!

- To prove a statement using a SAT solver:
  - Convert the statement to propositional logic
  - Negate it, and check for unsatisfiability
- Interesting corollary:
  - If the formula is SAT (implying the statement is false), the values that satisfy the statement are counter-examples

More details on SAT Solving (Book)

- Volume 4 Facsicle 6 of *The Art of Computer Programming*



More details on SAT/SMT Solving (Papers)

- Weissenbacher, Subramanyan, and Malik, Boolean Satisfiability Solvers: Techniques and Extensions
- Barrett, Sebastinani, Sheshia and Tinelli, Satisfiability Modulo Theories
- Niklas Eén, Niklas Sörensson, An Extensible SAT Solver

Quantifiers, Theories

- Propositional logic can be extended with quantifiers
  - $\exists$, existential quantifier
  - $\forall$, universal quantifier
  - This is First-order Logic (FOL)
  - FOL is undecidable in general
- Both propositional logic and first-order logic are still boolean
- Can be extended with theories:
  - Arithmetic: adds numbers, operators $+, -, \times$, associativity, commutativity, etc.
  - Functions: adds $f(x)$
  - Bitvectors: model variables containing n bits (where $n > 1$)

Satisfiability Modulo Theories (SMT)

- A SMT solver checks for satisfiability *in a theory*
  - Can think of statements as propositional logic + theory
- Allows construction of "richer" statements
  - Can formulate propositions over integers, reals, etc.
  - Can use operators like $+, -, \times$, etc.
- Example: $\forall_{x,y} x > y \implies x + 1 > y + 1$
  - True over integers ($\mathbb{Z}$)
  - False over machine integers/bitvectors

Decidability

- Some theories are decidable
  - Presburger arithmetic
- Most theories are undecidable
- However, some theories undecidable in general *are* decidable over quantifier-free fragments
- So, results of a SMT solver can be:
  - SAT
  - UNSAT
  - Don't know (or infinite loop)

Outline

Some SMT solvers

- Microsoft Z3
  - Used to be available online at rise4fun/Z3
  - Available in most Linux distributions
- CVC5
- Yices
- Many more...

The SMT-LIB language

- Common input/output language for most SMT solvers
  - Some solvers support their own language as well
- Lisp-like
- Documented at smtlib.org
- Allows easy switching between solvers
  - We will use Z3 for the most part

Encoding a problem for Z3 to solve

Let's encode $\neg R$ from the previous example:

```
(declare-fun B () Bool)
(declare-fun A () Bool)
(assert (not (or (and A B) (not A) (not B))))
(check-sat)
```

And we run it:

```
$ z3 p1.smt
unsat
```

Alternative: Python library

```
#!/usr/bin/env python3

from z3 import *

s = Solver()
A, B = Bools('A B')

R = Or(And(A, B), Or(Not(A), Not(B)))
notR = Not(R)

s.add(notR)
print(s.check())

print(s.sexpr())  # prints the SMTLIB code
```

See: Bjørner et al., Programming Z3, for a nice introduction to programming Z3 using Python.

SAT/SMT as constraint satisfaction

- The Constraint Satisfaction Problem seeks to find an assignment of values to variables subject to constraints
  - Each variable has a domain of values
  - Pick a variable, assign it a value, subject to constraints
  - If all variables can be assigned values, SAT else backtrack
- For SAT in propositional logic:
  - Two values, True and False
  - Constraint: formula must evaluate to true

Other Problems

- Dennis Yurichev's free book "SAT/SMT by Example" is a wonderful collection of examples
  - Minesweeper
  - Sudoku
  - Test case generation, etc.

Outline

SMT Solvers in Program Analysis

- Express program behaviour in some logic
- Express program property in that logic
- Check if the property holds (i.e. is valid)

Assertions

```
int min(int a, int b) {
  if(a < b)
      return a;
  else
      return a;
}

int x, y, r;

r = min(x, y);

assert(r == x || r == y);
assert(r <= x && r <= y);
```

- These assertions test that `min` always returns the minimum of `x` and `y`
- But `assert` executes at runtime
- We will seek to prove statically:
  - $\forall_{x,y}(\min(x,y) = x \vee \min(x,y) = y) \wedge (\min(x,y) \leq x \wedge \min(x,y) \leq y)$
  - over all program paths

Checking the correctness of `min`

```
from z3 import *

s = Solver()
a, b, ret = Ints('a b ret')
ret = If(a < b, a, a)
#ret = If(a < b, a, b) # correct
cond = And(Or(ret == a, ret ==b), And(ret <= a, ret <= b))
s.add(Not(cond))
print(s.sexpr())

if s.check() == sat:
    print("Incorrect. Counterexample: ", s.model())
else:
    print("Correct")
```

Output:

```
 ...
Incorrect. Counterexample:  [b = 0, a = 1]
```

Proving Programs Equivalent

- If A is a source program and B is the compiled version, we would like to prove that $A = B$
  - This is called *translation validation*
  - What I've been doing for your submissions
- Undecidable, in general
- Not interesting only to compiler writers
  - If you take a piece of code and refactor it, did you break anything?

Test Case Generation

```
int min(int a, int b) {
  if(a < b)
      return a;
  else
      return a;
}

int x, y, r;

r = min(1, 3);
 ...
```

- The test case $(1, 3)$ is not sufficient to exercise all paths in the program
  - And it misses the bug!
- Can we find test cases to exercise all paths in the program?

Postscript

- SMT solvers are a marvellous piece of technology
- Learn to use one!