

# PHASE TRANSITION FOR GLAUBER DYNAMICS FOR INDEPENDENT SETS ON REGULAR TREES\*

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**Abstract.** We study the effect of boundary conditions on the relaxation time (i.e., inverse spectral gap) of the Glauber dynamics for the hard-core model on the tree. The hard-core model is defined on the set of independent sets weighted by a parameter  $\lambda$ , called the activity or fugacity. The Glauber dynamics is the Markov chain that updates a randomly chosen vertex in each step. On the infinite tree with branching factor  $b$ , the hard-core model can be equivalently defined as a broadcasting process with a parameter  $\omega$  which is the positive solution to  $\lambda = \omega(1 + \omega)^b$ , and vertices are occupied with probability  $\omega/(1 + \omega)$  when their parent is unoccupied. This broadcasting process undergoes a phase transition between the so-called reconstruction and nonreconstruction regions at  $\omega_r \approx \ln b/b$ . Reconstruction has been of considerable interest recently since it appears to be intimately connected to the efficiency of local algorithms on locally tree-like graphs, such as sparse random graphs. In this paper we show that the relaxation time of the Glauber dynamics on regular trees  $T_h$  of height  $h$  with branching factor  $b$  and  $n$  vertices undergoes a phase transition around the reconstruction threshold. In particular, we construct a boundary condition for which the relaxation time slows down at the reconstruction threshold. More precisely, for any  $\omega \leq \ln b/b$ , for  $T_h$  with any boundary condition, the relaxation time is  $\Omega(n)$  and  $O(n^{1+o_b(1)})$ . In contrast, above the reconstruction threshold we show that for every  $\delta > 0$ , for  $\omega = (1 + \delta) \ln b/b$ , the relaxation time on  $T_h$  with any boundary condition is  $O(n^{1+\delta+o_b(1)})$ , and we construct a boundary condition where the relaxation time is  $\Omega(n^{1+\delta/2-o_b(1)})$ . To prove this lower bound in the reconstruction region we introduce a general technique that transforms a reconstruction algorithm into a set with poor conductance.

**Key words.** phase transition, mixing time, Glauber dynamics, Markov chain Monte Carlo hard-core model, independent sets

**1. Introduction.** There has been much recent interest in possible connections between equilibrium properties of statistical physics models and efficiency of local Markov chains for studying these models (see, e.g., [3, 10, 22, 23, 24, 34]). In this paper we study the hard-core model and establish new connections between the so-called reconstruction threshold in statistical physics with the convergence time of the single-site Markov chain known as the Glauber dynamics.

The hard-core model is studied in statistical physics as model of a lattice gas (see, e.g., Sokal [33]), and in operations research as a model of communication network

(see Kelly [17]). It is a natural combinatorial problem, corresponding to counting and randomly sampling weighted independent sets of an input graph  $G = (V, E)$ . Let  $\Omega = \Omega(G)$  denote the set of independent sets of  $G$ . Each set is weighted by an activity (or fugacity)  $\lambda > 0$ . For  $\sigma \in \Omega$ , its weight is  $w(\sigma) = \lambda^{|\sigma|}$ , where  $|\sigma|$  is the number of vertices in the set  $\sigma$ . The Gibbs measure is defined over  $\Omega$  as  $\mu(\sigma) = w(\sigma)/Z$ , where  $Z = \sum_{\sigma \in \Omega} w(\sigma)$  is the partition function.

This paper studies the hard-core model on trees, in some cases with a boundary condition. Let  $T_h$  denote the complete tree of height  $h$  with branching factor  $b$ . For concreteness we are assuming the root has  $b$  children, but our results, of course, easily extend to allow  $b + 1$  children for the root, the so-called Bethe lattice. Let  $n$  denote the number of vertices in  $T_h$ , and let  $L$  denote the leaves of the tree. A boundary condition is an assignment  $\Gamma$  to the leaves, where in the case of the hard-core model,  $\Gamma$  specifies a subset of the leaves  $L$  that are in the independent set. Then, let  $\Omega_\Gamma = \{\sigma \in \Omega : \sigma(L) = \Gamma\}$  be the set of independent sets of  $T_h$  that are consistent with  $\Gamma$ , and the Gibbs measure  $\mu_{h,\Gamma}$  is defined with respect to  $\Omega_\Gamma$ , i.e., it is the projection of  $\mu$  onto  $\Omega_\Gamma$ .

The (heat bath) Glauber dynamics is a discrete time Markov chain  $(X_t)$  for sampling from the Gibbs distribution  $\mu$  for a given graph  $G = (V, E)$  and activity  $\lambda$ . We view  $\Omega \subset \{0, 1\}^V$ , where for  $X_t \in \Omega$ ,  $X_t(v) = 1$  iff  $v$  is in the independent set. The transitions  $X_t \rightarrow X_{t+1}$  of the Glauber dynamics are defined as follows:

- Choose a vertex  $v$  uniformly at random.
- For all  $w \neq v$  set  $X_{t+1}(w) = X_t(w)$ .
- If all the neighbors of  $v$  are unoccupied, set  $X_{t+1}(v) = 1$  with probability  $\lambda/(1 + \lambda)$ , otherwise set  $X_{t+1}(v) = 0$ .

When a boundary condition  $\Gamma$  is specified, the state space is restricted to  $\Omega_\Gamma$ . For the case of the complete tree  $T_h$  (possibly with a boundary condition  $\Gamma$ ) it is straightforward to verify that for every  $\lambda > 0$  the Glauber dynamics is ergodic with unique stationary distribution  $\mu_h$  (or  $\mu_{h,\Gamma}$  when a boundary condition is specified). Thus, the Glauber dynamics is a natural algorithmic process for sampling from the Gibbs distribution. We study the relaxation time of the dynamics, which is defined as the inverse of the spectral gap of the transition matrix. See section 2 for a more detailed definition of the relaxation time.

The Gibbs distribution describes the equilibrium state of the system, and the Glauber dynamics is a model of how the physical system reaches equilibrium [14, 22]. Thus, it is interesting to understand connections between properties of the equilibrium state (i.e., the Gibbs distribution) and properties of how the system reaches equilibrium (i.e., the Glauber dynamics). Models from statistical physics are designed to study phase transitions in the equilibrium state. A phase transition is said to occur when a small change in the microscopic parameters of the system (in the case of the hard-core model that corresponds to  $\lambda$ ) causes a dramatic change in the macroscopic properties of the system.

A well-studied phase transition is the uniqueness/nonuniqueness of infinite volume Gibbs distributions. This phase transition corresponds to whether there exists a sequence of boundary conditions for which, roughly speaking, the root is “influenced” by the leaves in the limit  $h \rightarrow \infty$ . For the hard-core model on the complete tree, Kelly [17] showed that the uniqueness threshold is at  $\lambda_u = b^b/(b - 1)^{b+1}$  (namely, uniqueness holds iff  $\lambda < \lambda_u$ ).

There are interesting connections between the uniqueness threshold  $\lambda_u$  and the efficiency of algorithms on general graphs. In particular, Weitz [35] showed a deterministic fully polynomial approximation scheme to estimate the partition function

for any graph with constant maximum degree  $b$  for activities  $\lambda < \lambda_u$ . Sly [31] showed that it is NP-hard (unless  $NP = RP$ ) to approximate the partition function for activities  $\lambda$  satisfying  $\lambda_u < \lambda < \lambda_u + \epsilon_b$  for some small constant  $\epsilon_b$ . Sly's inapproximability result was extended to all  $\lambda > \lambda_u$  [11, 12, 32, 13].

We are interested in the phase transition for reconstruction/nonreconstruction. This corresponds to whether a “typical” boundary influences the root in the limit  $h \rightarrow \infty$ , whereas uniqueness/nonuniqueness considered the worst boundary condition. To construct a typical boundary, we consider the independent set on the leaves of  $T_h$  generated by the following broadcast process. This process constructs an independent set  $\sigma$  on the infinite tree in a top-down manner. Let  $\omega$  be the real positive solution of  $\lambda = \omega(1 + \omega)^b$ . Consider the infinite complete tree with branching factor  $b$ , and construct  $\sigma$  as follows. We first include the root  $r$  in  $\sigma$  with probability  $\omega/(1 + \omega)$  and exclude it with probability  $1/(1 + \omega)$ . Then for each vertex  $v$ , once the state of its parent  $p(v)$  is determined, if  $p(v) \notin \sigma$ , then we add  $v$  into  $\sigma$  with probability  $\omega/(1 + \omega)$  and leave it out with probability  $1/(1 + \omega)$ ; if  $p(v) \in \sigma$ , then we leave  $v$  out of  $\sigma$ . Let  $\sigma_h$  denote the set of leaves of  $T_h$  in  $\sigma$ .

Reconstruction addresses whether  $\sigma_h$  (in expectation) influences the configuration at the root  $r$  in the limit  $h \rightarrow \infty$ . That is, we first generate  $\sigma$  using the above broadcasting process. Then we fix  $\sigma_h$  on the leaves of  $T_h$  and resample a configuration  $\tau$  on  $T_h$  from the Gibbs distribution  $\mu_{h,\Gamma}$  with boundary condition  $\Gamma = \sigma_h$ . Of course, for finite  $h$ , the configuration at the root  $r$  in  $\tau$  has a bias to the initial configuration  $\sigma(r)$ . Nonreconstruction is said to hold if the root is unbiased in expectation in the limit  $h \rightarrow \infty$ . More precisely, reconstruction holds iff

$$(1.1) \quad \lim_{h \rightarrow \infty} \mathbf{E}_{\sigma \sim \nu_h} \left[ \left| \mu_{h,\sigma_h}(r \in \tau) - \frac{\omega}{1 + \omega} \right| \right] > 0.$$

There are many other equivalent conditions to the above definition of reconstruction; see Mossel [26] for a more extensive survey.

We refer to the reconstruction threshold as the critical  $\omega_r$  such that for all  $\omega < \omega_r$  nonreconstruction holds and for all  $\omega > \omega_r$  reconstruction holds. The existence of the reconstruction threshold follows from Mossel [27, Proposition 20], and by recent work of Bhatnagar, Sly, and Tetali [4] and Brightwell and Winkler [6], it is known that  $\omega_r = (\ln b + (1 + o(1)) \ln \ln b)/b$ .

Our interest in the reconstruction threshold is its apparent connections to the threshold for the efficiency of certain local algorithms on locally tree-like graphs, such as sparse random graphs  $G(n, c/n)$  for constant  $c > 1$ , planar graphs, and trees. Recent work on colorings suggests connections between the mixing time of the Glauber dynamics on trees and the reconstruction threshold; see [5, 30, 34]. Moreover, for colorings and independent sets, the reconstruction threshold on the tree is believed to be intimately connected to the threshold for the efficiency of local algorithms. The evidence in support of that belief is that the geometry of the space of solutions on sparse random graphs appears to change dramatically near (and possibly at) the reconstruction threshold; see [1, 15, 18, 25]. The results of [16] for the Glauber dynamics of colorings on planar graphs suggest that the reconstruction threshold may have connections to the mixing time of the Glauber dynamics on planar graphs. Moreover, an independent work of Coja-Oghlan and Efthymiou [7] proves related clustering results on random graphs that appear to occur at the reconstruction threshold and which imply an exponential slowdown in the mixing time of the Glauber dynamics. In addition, reconstruction for the Ising and Potts models has applications in phylogenetics [9].

In this paper we are interested in establishing more detailed connections between the reconstruction threshold and the relaxation time of the Glauber dynamics for trees. Berger et al. [3] proved that for the tree  $T_h$  with boundary condition  $\Gamma$  such that  $\mu_{h,\Gamma} = \nu_h$ ,  $O(n)$  relaxation time for all  $h$  implies nonreconstruction. For the Ising model and colorings the boundary condition is empty, i.e.,  $\nu_h$  corresponds to the free boundary condition. Hence, for these models, the result of [3] says that reconstruction implies relaxation time  $\omega(n)$ . For the hard-core model it is not clear if there is a boundary condition  $\Gamma$  for the finite tree which has the same measure as the broadcasting process, i.e.,  $\mu_{h,\Gamma} = \nu_h$ . This is discussed further in section 3.

It was recently established for the Ising model [3, 23, 8] and for  $k$ -colorings [34] that on the tree  $T_h$  with free boundary condition, the relaxation is  $O(n)$  in the non-reconstruction region and there is a slowdown in the reconstruction region. Our starting point was to address whether a similar phenomenon occurs in the hard-core model. Martinelli, Sinclair, and Weitz [24] showed that for the hard-core model on  $T_h$  with free boundary condition the relaxation time is  $O(n)$  for all  $\lambda$  (and the mixing time is  $O(n \log n)$ ). Hence, for the hard-core model, unlike in the Ising and colorings models, the Glauber dynamics on the tree with free boundary condition does not have connections to the reconstruction threshold. Our interest is whether there is a boundary condition for which there is such a connection.

We prove there is a connection by constructing a boundary condition for which the relaxation time slows down at the reconstruction threshold. Here is the formal statement of our results.

**THEOREM 1.1.** *For the Glauber dynamics on the hard-core model with activity  $\lambda = \omega(1 + \omega)^b$  on the complete tree  $T_h$  with  $n$  vertices, height  $h$ , and branching factor  $b$ , the following hold:*

1. *For all  $\omega \leq \ln b/b$ , for every boundary condition,*

$$\Omega(n) \leq T_{\text{relax}} \leq O(n^{1+o_b(1)}).$$

2. *For all  $\delta > 0$  and  $\omega = (1 + \delta) \ln b/b$ ,*
  - (a) *for every boundary condition,*

$$T_{\text{relax}} \leq O(n^{1+\delta+o_b(1)});$$

- (b) *there exists a sequence of boundary conditions for all  $h \rightarrow \infty$  such that*

$$T_{\text{relax}} = \Omega(n^{1+\delta/2-o_b(1)}).$$

*Remark 1.* More precisely, we show that there is a function  $g(b) = O(\ln \ln b / \ln b) = o(1)$  such that for every  $b$ , the lower bound in part 2(b) is  $\Omega(n^{1+\delta/2-g(b)})$ , and there is a function  $f(b) = O((\ln \ln b)^2 / \ln b) = o(1)$  such that for every  $b$ , the upper bound in part 1 is  $O(n^{1+f(b)})$  and in part 2(a) is  $O(n^{1+\delta+f(b)})$ .

The upper bound improves upon Martinelli, Sinclair, and Weitz, [24], who showed  $O(n)$  relaxation time (and  $O(n \log n)$  mixing time) for  $\lambda < 1/(\sqrt{b}-1)$  for all boundary conditions. Note that  $\lambda = 1/\sqrt{b}$  is roughly equivalent to  $\omega \approx \frac{1}{2} \ln b/b$ , which is below the reconstruction threshold. Our main result extends the fast mixing up to the reconstruction threshold and shows the slowdown beyond the reconstruction threshold. Our lower bound in the reconstruction region uses a general approach. An algorithm showing reconstruction is used to construct a set with poor conductance, which implies a lower bound on the relaxation time. This framework captures the proof approach used in [34].

We were facing two major difficulties: one is to identify a proper subset of the state space with poor conductance, such that the corresponding conductance bound closely matches the relaxation time of the Glauber dynamics. Also, the conductance of such a subset should be sensitive to the boundary conditions, as we already know that the Glauber dynamics is rapid mixing under properly chosen boundary conditions (see, e.g., [24]). The other difficulty, once we realize that the relaxation time of the Glauber dynamics can be nontrivially lower bounded under a nonuniform hard-core model (see section 3 for details), is to prove that when reconstruction happens, such a nonuniform model can be approximated (in the measure sense), by an appropriate sequence of boundary conditions. As a result, then we are able to show part 2(b) via a conductance argument.

In section 2 we formally define various terms and present the basic tools used in our proofs. The lower bound (part 2(b) of Theorem 1.1) is presented in sections 3, 4, and 5. Section 3 outlines the approach. We then prove an analogue of Theorem 1.1 in section 4 for the broadcasting model and use it in section 5 to prove part 2(b) of Theorem 1.1. The argument for the upper bounds stated in Theorem 1.1 is presented in section 7.

## 2. Background.

**2.1. Spectral gap.** Let  $P(\cdot, \cdot)$  denote the transition matrix of the Glauber dynamics. Let  $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_{|\Omega|}$  be the eigenvalues of the transition matrix  $P$ . The spectral gap  $c_{gap}$  is defined as  $1 - \gamma$ , where  $\gamma = \max\{\gamma_2, |\gamma_{|\Omega|}|\}$  denotes the second-largest eigenvalue in absolute value. The relaxation time  $T_{relax}$  of the Markov chain is then defined as  $c_{gap}^{-1}$ , the inverse of the spectral gap. Relaxation time is an important measure of the convergence rate of a Markov chain (see, e.g., Chapter 12 in [20]).

**2.2. Mixing time.** Another common measure of the convergence time is the mixing time  $T_{mix}$ . The total variation distance at time  $t$  from initial state  $\sigma$  is defined as

$$\|P^t(\sigma, \cdot) - \pi\|_{TV} := \frac{1}{2} \sum_{\eta \in \Omega} |P^t(\sigma, \eta) - \pi(\eta)|.$$

The mixing time  $T_{mix}$  for a Markov chain is then defined as

$$T_{mix} = \min_{t > 0} \left\{ \max_{\sigma \in \Omega} \{ \|P^t(\sigma, \cdot) - \pi\|_{TV} \} \leq 1/2e \right\}$$

as the number of steps, from the worst initial state, to reach within total variation distance  $\leq 1/2e$  of the stationary distribution  $\pi$ . In Theorem 1.1 we stated our main results in terms of relaxation time. An upper bound on the relaxation time implies the following bound on the mixing time (cf. Theorem 12.3 in [20]):

$$T_{mix} = O \left( T_{relax} \ln \frac{1}{\min_{\sigma \in \Omega} \{\pi(\sigma)\}} \right).$$

For the case of the complete tree one can potentially obtain tighter upper bounds on the mixing time by using the approach of [23, Theorem 5.7], as was done in [34, section 8], to use the spectral gap to first bound the log-Sobolev constant.

It is an elementary fact that a lower bound on the relaxation time implies the same lower bound on the mixing time (see, e.g., Theorem 12.4 in [20] for the following bound),

$$(2.1) \quad T_{relax} \leq T_{mix} + 1.$$

**2.3. Conductance.** To lower bound the relaxation time we analyze conductance. The conductance of a Markov chain with state space  $\Omega$  and transition matrix  $P$  is given by  $\Phi = \min_{S \subseteq \Omega} \{\Phi_S\}$ , where  $\Phi_S$  is the conductance of a specific set  $S \subseteq \Omega$  defined as

$$\Phi_S = \frac{\sum_{\sigma \in S} \sum_{\eta \in \bar{S}} \pi(\sigma) P(\sigma, \eta)}{\pi(S) \pi(\bar{S})}.$$

A general way to find a good upper bound on the conductance is to find a set  $S$  such that the probability of “escaping” from  $S$  is relatively small. The well-known relationship between the relaxation time and the conductance was established in [19] and [29], and we will use the form  $T_{\text{relax}} = \Omega(1/\Phi)$  for proving the lower bounds.

**2.4. Coupling.** To upper bound the mixing time (and hence the relaxation time) we will use the coupling method. Given two copies  $(X_t)$  and  $(Y_t)$  of the Glauber dynamics, a coupling is a joint process  $(X_t, Y_t)$  such that the evolution of each component viewed in isolation is identical to the Glauber dynamics (cf. [20] for an introduction to the coupling technique). In many situations, the coupling lemma [2] (cf. [20, Theorem 5.2]) is used to upper bound the mixing time. It guarantees that if there is a coupling and time  $t > 0$ , so that for every pair  $(X_0, Y_0)$  of initial states,  $\Pr[X_t \neq Y_t \mid X_0, Y_0] \leq 1/2e$  under the coupling, then  $T_{\text{mix}} \leq t$ .

**3. Lower bound approach.** First note that the lower bound stated in part 1 of Theorem 1.1, namely,  $T_{\text{relax}} = \Omega(n)$ , is trivial for all  $\omega$ . For example, by considering the set  $S = \{\sigma \in \Omega : r \notin \sigma\}$  of independent sets which do not contain the root,  $\Phi(S) = \Omega(1/n)$  since we need to update  $r$  to leave  $S$ .

We begin by explaining the high-level idea of the nontrivial lower bound in part 2(b) of Theorem 1.1. To that end, we first analyze a variant of the hard-core model in which there are two different activities; the internal vertices have activity  $\lambda$ , and the leaves have activity  $\omega$ . The resulting Gibbs distribution is identical to the measure  $\nu_h$  defined in section 1 for the broadcasting process. Thus we refer to the following model also as the broadcasting model.

For the tree  $T_h = (V, E)$ , we look at the following equivalent definition of the distribution  $\nu_h$  over the set  $\Omega$  of independent sets of  $T_h$ . For  $\sigma \in \Omega$ , let

$$w'(\sigma) = \lambda^{|\sigma \cap V \setminus L|} \omega^{|\sigma \cap L|},$$

where  $L$  are the leaves of  $T_h$  and  $\omega$  is, as before, the positive solution to  $\omega(1 + \omega)^b = \lambda$ . Let  $\nu_h(\sigma) = w'(\sigma)/Z'$ , where  $Z' = \sum_{\sigma \in \Omega} w'(\sigma)$  is the partition function. By simple calculations, the following proposition holds.

**PROPOSITION 3.1.** *The measure  $\nu_h$  defined by the hard-core model with activity  $\lambda$  for internal vertices and  $\omega$  for leaves is identical to the measure defined by the broadcasting process.*

*Proof.* In fact, we just need to verify that in the hard-core model with activity  $\lambda$  for internal vertices and  $\omega$  for leaves, the probability  $p_v$  of a vertex  $v$  being occupied conditioned on its parent being unoccupied is  $\omega/(1 + \omega)$ . This can be proved by induction. The base case is  $v$  being a leaf, which is obviously true by the Markovian property of the Gibbs measure. If  $v$  is not a leaf, by induction, the probability  $p_v$  has to satisfy the equation

$$p_v = (1 - p_v) \frac{\lambda}{(1 + \omega)^b},$$

which solves to  $p_v = \omega/(1 + \omega)$ .  $\square$

The result of Berger et al. [3] mentioned in section 1 implies that in the reconstruction region, the relaxation time of the Glauber dynamics on the broadcasting model is  $\omega(n)$ . We will prove a stronger result, analogous to the desired lower bound for part 2(b) of Theorem 1.1.

**THEOREM 3.2.** *For all  $\delta > 0$ , the Glauber dynamics for the broadcasting model on the complete tree  $T_h$  with  $n$  vertices, branching factor  $b$ , and  $w = (1 + \delta) \ln b/b$  satisfies the following:*

$$T_{\text{relax}} = \Omega(n^{1+\delta/2-o_b(1)}),$$

where the  $o_b(1)$  function is  $O(\ln \ln b / \ln b)$ .

*Remark 2.* We can show a similar upper bound on the relaxation time for the Glauber dynamics in this setting as in Theorem 1.1. Moreover, we can show the same upper bound for the mixing time by establishing a tight bound between the inverse log-Sobolev constant and the relaxation time, as was done for colorings in Tetali et al. [34].

We will prove Theorem 3.2 via a general method that relates any reconstruction algorithm (or function) with the conductance of the Glauber dynamics. A *reconstruction algorithm* is a function  $A : \Omega(L) \rightarrow \{0, 1\}$  (ideally efficiently computable) such that  $A(\sigma_h)$  and  $\sigma(r)$  are positively correlated. Basically, algorithm  $A$  takes the configurations at leaves  $L$  as the input and tries to compute the configuration at the root. When the context is clear, we write  $A(\sigma)$  instead of  $A(\sigma_h)$ . Under the Gibbs measure  $\nu_h$ , the *effectiveness* of  $A$  is the following measure of the covariance between algorithm  $A$ 's output and the marginal at the root of the actual measure:

$$r_{h,A} = \min_{x \in \{0,1\}} [\nu_h(A(\sigma) = \sigma(r) = x) - \nu_h(A(\sigma) = x)\nu_h(\sigma(r) = x)].$$

If it is the case that

$$\liminf_{h \rightarrow \infty} r_{h,A} = c_0 > 0$$

for some positive constant  $c_0$  depending only on  $\omega$  and  $b$ , then we say that  $A$  is an *effective reconstruction algorithm*. In words, an effective algorithm, is able to recover the spin at the root, from the information at the leaves, with a nontrivial success, when  $h \rightarrow \infty$ . Notice that reconstruction (see (1.1)) is a necessary condition for any reconstruction algorithm to be effective, since

$$\begin{aligned} \mathbf{E}_{\sigma \sim \nu_h} \left[ \left| \mu_{h,\sigma_h}(r \in \tau) - \frac{\omega}{1 + \omega} \right| \right] &\geq \mathbf{E}_{\sigma \sim \nu_h} [(\mu_{h,\sigma_h}(r \in \tau) - \nu_h(r \in \sigma)) \mathbf{1}(A(\sigma) = 1)] \\ &\geq r_{h,A}, \end{aligned}$$

where  $\mathbf{1}()$  is the indicator function. We define the *sensitivity* of  $A$ , for the configuration  $\sigma \in \Omega(T_h)$ , as the fraction of vertices  $v$  such that switching the spin at  $v$  in  $\sigma$  changes the final result of  $A$ . More precisely, let  $\sigma^v$  be the configuration obtained from changing  $\sigma$  at  $v$ . Define the sensitivity as

$$S_A(\sigma) = \frac{1}{n} \#\{v \in L : A(\sigma^v) \neq A(\sigma)\}.$$

The *average sensitivity* (with respect to the root being occupied)  $\bar{S}_A$  is hence defined as

$$\bar{S}_A = \mathbf{E}_{\sigma \sim \nu_h} [S_A(\sigma) \mathbf{1}(A(\sigma) = 1)].$$

It is fine to define the average sensitivity without the indicator function, which only affects a constant factor in the analysis. We are doing so to simplify some of the statements and proofs.

Typically when one proves reconstruction, it is done by presenting an effective reconstruction algorithm. Using the following theorem, by further analyzing the sensitivity of the reconstruction algorithm, one obtains a lower bound on the relaxation time or mixing time of the Glauber dynamics.

**THEOREM 3.3.** *Suppose that  $A$  is an effective reconstruction algorithm. Then, the relaxation time  $T_{\text{relax}}$  of the Glauber dynamics satisfies  $T_{\text{relax}} = \Omega((\bar{S}_A)^{-1})$ .*

*Remark 3.* The above theorem can be generalized to any spin system. To illustrate the usefulness of this theorem, we note that the lower bound on the mixing time of the Glauber dynamics for  $k$ -colorings in the reconstruction region proved in [34] fits this conceptually appealing framework.

*Proof.* Throughout the proof let  $\nu := \nu_h$ . Consider the set  $U = \{\sigma : A(\sigma) = 1\}$ . Recall that  $P$  is the transition matrix of the Glauber dynamics. Then,

$$\begin{aligned} \Phi_U &= \frac{\sum_{\sigma \in U} \nu(\sigma) \sum_{w \in L} \sum_{\tau: \tau(w) \neq \sigma(w)} P(\sigma, \tau)}{\nu(U)(1 - \nu(U))} \\ &\leq \frac{\sum_{\sigma \in U} \nu(\sigma) S_A(\sigma)}{\nu(U)(1 - \nu(U))} \\ &\leq \frac{\bar{S}_A}{\nu(A(\sigma) = \sigma(r) = 1)\nu(A(\sigma) = \sigma(r) = 0)} \\ &\leq \frac{\bar{S}_A}{r_{h,A}^2} \quad \text{by the definition of } r_{h,A}. \end{aligned}$$

Because the algorithm is effective, we have that  $\liminf_{h \rightarrow \infty} r_{h,A} = c_0 > 0$  and hence for all  $h$  big enough,  $r_{h,A} > c_0/2$ . Therefore,  $\Phi_U \leq (r_{h,A})^{-2} \bar{S}_A = O(\bar{S}_A)$ , and hence

$$T_{\text{relax}} = c_{\text{gap}}^{-1} \geq 1/\Phi_U = \Omega((\bar{S}_A)^{-1}),$$

which completes the proof of the theorem.  $\square$

To prove Theorem 3.2, we analyze the sensitivity of the reconstruction algorithm by Brightwell and Winkler [6, section 5], which yields the best known upper bounds on the reconstruction threshold. Our goal is to show that the average sensitivity of this algorithm is small. The analysis of the sensitivity of the Brightwell-Winkler (BW) algorithm, which then proves Theorem 3.2, is presented in section 4.

Our main objective remains constructing a sequence of “bad” boundary conditions under which the Glauber dynamics for the hard-core model slows in the reconstruction region. An initial approach is to simulate the nonuniform hard-core model on  $T$  by attaching the same tree  $T'$  (with boundary conditions) to all the leaves of a complete tree  $T$ , where  $T'$  is a (small) complete tree with some boundary condition such that the marginal of the root being occupied is  $\omega/(1 + \omega)$ . In this case, the resulting measure projected onto  $T$  is the same as the one in the broadcasting model, and hence we can apply the same approach to upper bound the conductance of the dynamics on this new augmented tree. However, from a cardinality argument, it is not the case that for every  $\omega$  there exists a complete tree of finite height with some boundary condition such that the marginal probability of the root being occupied equals  $\omega/(1 + \omega)$ . Alternatively, we give a constructive way to find boundary conditions that approximate the desired marginal probability relatively accurately. This is done in section 5.

Finally, at the end of section 5 we argue that since the error is shrinking very fast from the bottom level under our construction of boundary conditions, we can again analyze the sensitivity of the BW algorithm starting from just a few levels above the leaves. This approach yields the lower bound stated in part 2(b) of Theorem 1.1.

**4. Lower bound for broadcasting: Proof of Theorem 3.2.** Throughout this section we are working with the tree  $T_h = (V, E)$ , which is the complete tree of height  $h$  and branching factor  $b$ . We denote  $L$  as the leaves of  $T_h$ , and for  $v \in V$  let  $N(v)$  denote the children of  $v$ . We will focus on the broadcasting model  $\nu_h$  (defined in section 3), where each independent set  $\sigma$  of tree  $T_h$  is weighted by  $\lambda^{|\sigma \cap V \setminus L|} \omega^{|\sigma \cap L|}$ . Recall that  $\lambda = \omega(1+\omega)^b$ . For simplicity, we identify  $\sigma$  with its characteristic function. We use the following function definition for  $\sigma$ :  $\sigma(v) = 1$  if  $v \in \sigma$ , and  $\sigma(v) = 0$  if  $v \notin \sigma$ .

To prove Theorem 3.2 we analyze the average sensitivity of the following reconstruction algorithm used by Brightwell and Winkler [6], which we refer to as the BW algorithm. For any configuration  $\sigma \in \Omega$  as the input (or it suffices to have the assignment  $\sigma_h$  for the leaves), the algorithm works in the following bottom-up manner labeling each vertex starting from the leaves: a vertex  $v$  is labeled “occupied” if all its children  $N(v)$  are labeled “unoccupied”; otherwise,  $v$  is labeled “unoccupied” if at least one of its children  $N(v)$  is labeled “occupied.” The algorithm will output the labeling of the root as the final result. Formally, it can be described by the following deterministic recursion deciding the labeling of every vertex:

$$BW(\sigma_h, v) = \begin{cases} \sigma(v) & \text{if } v \in L, \\ 1 - \max_{w \in N(v)} BW(\sigma_h, w) & \text{otherwise.} \end{cases}$$

Finally, let  $BW(\sigma) = BW(\sigma_h, r)$ , where  $r$  is the root of the tree. Note that  $BW(\sigma)$  only depends on the initial configuration  $\sigma_h$  of the leaves. The algorithm is proved to be effective in [6] for all  $\omega = (1 + \delta) \ln b/b$ , where  $\delta > 0$ . Therefore, their algorithm can be used under our framework to lower bound the relaxation time.

In the BW algorithm, by definition, we have that the average sensitivity satisfies

$$(4.1) \quad \bar{S}_{BW} = O(n^{-1} \mathbf{E}_{\sigma \sim \nu_h} [\#\{z \in L : BW(\sigma) = 1 \text{ and } BW(\sigma^z) = 0\}]).$$

Due to the symmetry of the function  $BW(\sigma_h, v)$  and the measure  $\nu_h$ , the expectation can be further simplified as follows. Fix a leaf  $z^*$ ; we have that

$$(4.2) \quad \begin{aligned} \mathbf{E}_{\sigma} [\#\{z \in L : BW(\sigma) = 1 \text{ and } BW(\sigma^z) = 0\}] \\ = b^h \nu_h(BW(\sigma) = 1 \text{ and } BW(\sigma^{z^*}) = 0). \end{aligned}$$

Observe that for each vertex  $v$  and each configuration  $\sigma$ , if  $BW(\sigma_h, v) \neq BW(\sigma_h^{z^*}, v)$ , then  $z^*$  is a leaf on the subtree rooted at  $v$ , and moreover, for each child  $w$  of  $v$  which is not on the path from  $v$  to  $z^*$ ,  $BW(\sigma_h, w) = 0$ . This fact leads to the following lemma that we will use to upper bound the right-hand side of (4.2).

LEMMA 4.1. *Let  $z^*$  be a leaf of  $T_h$ , and let  $z^* = u_0, u_1, \dots, u_h = r$  be the path between  $z^*$  and the root of  $T_h$ . For each  $i > 0$ , let*

$$f_i = \nu_{i-1}(\sigma : BW(\sigma) = 0)$$

*denote the probability that for the broadcasting model on the complete tree of height  $i - 1$ , the BW algorithm outputs 0 for the root. Then,*

$$\nu_h(\text{BW}(\sigma) = 1 \text{ and } \text{BW}(\sigma^{z^*}) = 0) \leq \mathbf{E}_{\sigma \sim \nu_h} \left[ \prod_{i>0:\sigma(u_i)=0} (f_i)^{b-1} \right].$$

*Proof.* Fix a configuration  $\sigma \in \Omega$ , where  $\text{BW}(\sigma) = \text{BW}(\sigma_h, r) = 1$ . Let the path  $\mathcal{P}$  from  $z^*$  to the root  $r$  be  $u_0 = z^*, u_1, u_2, \dots, u_h = r$ . Let  $\hat{N}(u_i) = N(u_i) \setminus \{u_{i-1}\}$ . We want that  $\text{BW}(\sigma) = 1$  and  $\text{BW}(\sigma^{z^*}) = 0$ , i.e., by changing  $\sigma$  only at  $z^*$ , the output of the BW algorithm changes from occupied to unoccupied for the labeling of the root. Two necessary conditions for this to occur are the following. First, the output of the BW algorithm along the path  $\mathcal{P}$  alternates between occupied and unoccupied, i.e.,  $\sigma$  satisfies  $\text{BW}(\sigma_h, u_i) = 1 - \text{BW}(\sigma_h, u_{i-1})$  for all  $i \geq 1$ . Second, for all  $i \geq 1$ , for all children  $w \in \hat{N}(u_i)$ , we have  $\text{BW}(\sigma_h, w) = 0$ . These two conditions ensure that if the configuration at  $u_i$  changes, then the output of the BW algorithm will change for  $u_{i-1}$ . Hence,

$$(4.3) \quad \nu_h(\text{BW}(\sigma) = 1, \text{BW}(\sigma^{z^*}) = 0) \leq \nu_h(\sigma : \forall i > 0, w \in \hat{N}(u_i), \text{BW}(\sigma_h, w) = 0).$$

To calculate the probability that a random  $\sigma \sim \nu_h$  satisfies such conditions, it would be easier if we expose the configurations along the path  $\mathcal{P}$ . Let  $\sigma_{\mathcal{P}}$  be the projection of  $\sigma$  on the path  $\mathcal{P}$ . Conditioning on a configuration  $\sigma_{\mathcal{P}}$  on the path, the events  $\text{BW}(\sigma_h, w) = 0$  are independent for all  $w \in \bigcup_{i>0} \hat{N}(u_i)$ . Note that, given  $\sigma(u_i) = 0$ , we have for all  $w \in \hat{N}(u_i)$  that the conditional probability of  $\text{BW}(\sigma_h, w) = 0$  equals  $f_i$ . Therefore,

$$\begin{aligned} &\nu_h(\text{BW}(\sigma) = 1 \text{ and } \text{BW}(\sigma^{z^*}) = 0) \\ &\leq \nu_h(\sigma : \forall i > 0, w \in \hat{N}(u_i), \text{BW}(\sigma_h, w) = 0) \quad \text{by (4.3)} \\ &= \sum_{\eta \in \{0,1\}^{|\mathcal{P}|}} \nu_h(\sigma : \sigma_{\mathcal{P}} = \eta) \prod_{i=1}^h \prod_{w \in \hat{N}(u_i)} \Pr_{\sigma \sim \nu_h} [\text{BW}(\sigma_h, w) = 0 \mid \sigma(u_i) = \eta(u_i)] \\ &\leq \sum_{\eta \in \{0,1\}^{|\mathcal{P}|}} \nu_h(\sigma : \sigma_{\mathcal{P}} = \eta) \prod_{i>0:\sigma(u_i)=0} \prod_{w \in \hat{N}(u_i)} \Pr_{\sigma \sim \nu_h} [\text{BW}(\sigma_h, w) = 0 \mid \sigma(u_i) = 0] \\ &= \sum_{\eta \in \{0,1\}^{|\mathcal{P}|}} \nu_h(\sigma : \sigma_{\mathcal{P}} = \eta) \prod_{i>0:\sigma(u_i)=0} \prod_{w \in \hat{N}(u_i)} f_i \\ &= \sum_{\eta \in \{0,1\}^{|\mathcal{P}|}} \nu_h(\sigma : \sigma_{\mathcal{P}} = \eta) \prod_{i>0:\eta(u_i)=0} (f_i)^{b-1} \\ &= \mathbf{E}_{\sigma \sim \nu_h} \left[ \prod_{i>0:\sigma(u_i)=0} (f_i)^{b-1} \right]. \quad \square \end{aligned}$$

To use Lemma 4.1, we derive the following uniform upper bound on the probability  $f_i$ , for all  $i$ . Note that, since our bounds are asymptotic, we will always assume that the degree  $b$  is large enough with respect to  $\delta$  to make our proofs simpler. In particular, for  $\omega = (1 + \delta) \ln b/b$  and  $\lambda = \omega(1 + \omega)^b$ , let

$$(4.4) \quad b_0(\delta) := \min \left\{ b' \geq 10^4 : \exp \left( \frac{2(1.01)(\omega b)^2}{\lambda} \right) \leq 1.01 \forall b > b' \right\}.$$

(Note that the extra factor of 2 in the exponential is not needed in the proof of Lemma 4.2, but is convenient in section 5 for the proof of Proposition 5.7.) Note that  $b_0(\delta)$  is well-defined since for any fixed  $\delta$ ,

$$\lim_{b \rightarrow \infty} \exp\left(\frac{2(1.01)(\omega b)^2}{\lambda}\right) = \lim_{b \rightarrow \infty} \exp\left(\frac{2(1.01) \ln b}{b^\delta}\right) < 1.01.$$

LEMMA 4.2. For all  $\delta > 0$ , all  $b > b_0(\delta)$ , and  $i \geq 1$ , setting  $\omega = (1 + \delta) \ln b/b$ , we have

$$f_i \leq \frac{(1.01)^{1/b}}{1 + \omega}.$$

*Proof.* We will prove the lemma by induction. We first derive the recurrence of  $f_i$  for each  $i$ . For the base case  $i = 1$ , by the definition of the broadcasting model,

$$f_1 = \frac{1}{1 + \omega}.$$

When  $i = 2$ ,  $f_2$  is the probability that the complete tree of height two has at least one child that is occupied. This requires us to first unoccupy the root with probability  $1/(1 + \omega)$  and then have at least one child occupied. Therefore,

$$f_2 = \frac{1}{1 + \omega} \left(1 - \left(\frac{1}{1 + \omega}\right)^b\right).$$

Generally, one can see the recurrence holds for  $f_{i+1}$  by looking into two cases of  $\sigma$  sampled from distribution  $\nu_h$ : Occupy the root  $r$  with probability  $\omega/(1 + \omega)$  in  $\sigma$  and then calculate the conditional probability of having at least one child that is labeled as 1 (occupied) in the BW algorithm. This is the complement of the event that all the children of  $r$  having at least one of their own children reconstruct to occupied in the BW algorithm, given the fact that all children of  $r$  are fixed to unoccupied in  $\sigma$ . The probability of this event happening equals  $(1 - (f_{i-1})^b)^b$ . The second case occurs when we do not occupy the root with probability  $1/(1 + \omega)$  in  $\sigma$  and then the event occurs that at least one child is labeled 1 in the BW algorithm. Thus,

$$(4.5) \quad f_{i+1} = \frac{\omega}{1 + \omega} \left(1 - (1 - (f_{i-1})^b)^b\right) + \frac{1}{1 + \omega} (1 - (f_i)^b).$$

Therefore, we have

$$\begin{aligned} f_{i+1} &\leq \frac{\omega}{1 + \omega} \left(1 - (1 - (f_{i-1})^b)^b\right) + \frac{1}{1 + \omega} && \text{by (4.5)} \\ &\leq \frac{\omega}{1 + \omega} \left(1 - \left(1 - \frac{1.01\omega}{\lambda}\right)^b\right) \\ &\quad + \frac{1}{1 + \omega} && \text{by the induction hypothesis applied to } f_{i-1} \\ &\leq \frac{1 + \frac{1.01\omega^2 b}{\lambda}}{1 + \omega} && \text{since } (1 - t)^b \geq 1 - tb \text{ for } t < 1 \\ &\leq \frac{\exp(1.01\omega^2 b/\lambda)}{1 + \omega} && \text{since } (1 + t) \leq e^t \\ &\leq \frac{(1.01)^{1/b}}{1 + \omega}. && \text{from the definition of } b_0(\delta) \text{ in (4.4). } \quad \square \end{aligned}$$

Now, we combine Lemmas 4.1 and 4.2 to prove Theorem 3.2.

*Proof of Theorem 3.2.* Fix a leaf  $z^*$  of  $T_h$ , and let  $\mathcal{P}$  be the path  $z^* = u_0, u_1, \dots, u_h = r$  between  $z^*$  and the root of  $T_h$ . We upper bound the average sensitivity of the BW algorithm in the following way:

$$\begin{aligned} \bar{S}_{\text{BW}} &= O(\nu_h(\text{BW}(\sigma) = 1 \text{ and } \text{BW}(\sigma^{z^*}) = 0)) && \text{by (4.1) and (4.2)} \\ &= O\left(\mathbf{E}_{\sigma \sim \nu_h} \left[ \left( \frac{1.01\omega(1+\omega)}{\lambda} \right)^{\#\{i:\sigma(u_i)=0\}} \right]\right) && \text{by Lemmas 4.1 and 4.2.} \end{aligned}$$

In this expectation, the number of unoccupied vertices in  $\mathcal{P}$  can be trivially lower bounded by  $h/2$ , since it is impossible to have two consecutive occupied vertices in  $\mathcal{P}$ . Therefore, the above expectation can be easily bounded by  $O^*(n^{-(1+\delta)/2})$ . This is not good enough in our case: to establish the existence of a phase transition we need a bound of the form  $O^*(n^{-(1+\delta/2)})$ . This improved bound will be a consequence of the following lemma.

LEMMA 4.3. *For all  $\delta > 0$ , all  $b > b_0(\delta)$ , setting  $\omega = (1 + \delta) \ln b/b$ , we have*

$$\mathbf{E}_{\sigma \sim \nu_h} \left[ \left( \frac{1.01\omega(1+\omega)}{\lambda} \right)^{\#\{i:\sigma(u_i)=0\}} \right] = O\left( \left[ \frac{1.01\omega}{\lambda^{1/2}} \right]^h \right).$$

Lemma 4.3 is proved in section 6. Then, by the fact that the height of the tree is  $h = \log_b n$ , we have, for  $\delta > 0$  and all  $b > b_0(\delta)$ ,

$$\bar{S}_{\text{BW}} = O\left( \left[ \frac{1.01\omega}{\lambda^{1/2}} \right]^h \right) = O\left( n^{-\left[ 1 + \frac{\ln(\lambda/(1.01\omega b)^2)}{2 \ln b} \right]} \right),$$

where  $\omega = (1 + \delta) \ln b/b$ . Now, from the fact in [6, section 5] that the BW algorithm is effective for all  $\delta > 0$ ,  $\omega > (1 + \delta) \ln b/b$  and  $b > b_0(\delta)$  (a similar statement is proved later in our paper in Proposition 5.7.), Theorem 3.3 applies, and the conclusion follows for the relaxation time, allowing us to conclude that for  $\delta > 0$ , and  $\omega = (1 + \delta) \ln b/b$ , for all  $b \geq b_0(\delta)$ ,

$$T_{\text{relax}} = \Omega(n^d), \quad \text{where } d = \left( 1 + \frac{\ln(\lambda/(1.01\omega b)^2)}{2 \ln b} \right).$$

Theorem 3.2 is a simple corollary by noticing that  $d = 1 + \delta/2 - O(\frac{\ln \ln b}{\ln b})$ . Note that when  $b < b_0(\delta)$ , our bound is trivial.  $\square$

**5. “Bad” boundary conditions: Proof of Theorem 1.1, 2(b).** First, we will show that for any  $\omega$ , there exists a sequence of boundary conditions, denoted as  $\Gamma_\omega := \{\Gamma_i\}_{i>0}$ , one for each complete tree of height  $i > 0$ , such that if  $i \rightarrow \infty$ , the probability of the root being occupied converges to  $\frac{\omega}{1+\omega}$ . Later in this section we will exploit such a construction to attain in full the conclusion of part 2(b) of Theorem 1.1.

As a first observation, note that the Gibbs measure for the hard-core model on  $T_i$  with boundary condition  $\Gamma$ , is the same as the Gibbs measure for the hard-core model (with the same activity  $\lambda$ ) on the tree  $T$  obtained from  $T_i$  by deleting all the leaves as well as the parent of each (occupied) leaf  $v \in \Gamma$ . It will be convenient to work directly with such “trimmed” trees, rather than the complete tree with boundary condition. Having this in mind, our construction will be inductive in the following way. We will define a sequence of (trimmed) trees  $\{(L_i, U_i)\}_{i \geq 0}$  such that  $L_{i+1}$  is comprised of  $s_{i+1}$

copies of  $U_i$  and  $b - s_{i+1}$  copies of  $L_i$  with  $\{s_i\}_{i \geq 1}$  properly chosen. Similarly,  $U_{i+1}$  is comprised of  $t_{i+1}$  copies of  $U_i$  and  $b - t_{i+1}$  copies of  $L_i$ , with  $\{t_i\}_{i \geq 1}$  properly chosen.

We will show that for either  $T_i^* = L_i$  or  $T_i^* = U_i$ , it is the case that the  $Q$ -value, defined as

$$Q(T_i^*) = \frac{\mu_{T_i^*}(\sigma(r) = 1)}{\omega \mu_{T_i^*}(\sigma(r) = 0)},$$

where  $\mu_{T_i^*}(\cdot)$  is the hard-core measure on the trimmed tree  $T_i^*$ , satisfies  $Q(T_i^*) \rightarrow 1$ . Note that if  $Q(T_i^*) = 1$ , then the probability of the root being occupied is  $\omega/(1 + \omega)$  as desired. To attain this, we will construct  $L_i$  and  $U_i$  in such a way that  $Q(U_i) \geq 1$  and  $Q(L_i) \leq 1$ .

The recursion for  $Q(L_{i+1})$  can be derived easily as

$$Q(L_{i+1}) = \frac{(1 + \omega)^b}{(1 + \omega Q(U_i))^{s_{i+1}} (1 + \omega Q(L_i))^{b - s_{i+1}}},$$

and a similar equation holds for  $Q(U_{i+1})$  by replacing  $s_{i+1}$  with  $t_{i+1}$ .

To keep the construction simple, we inductively define the appropriate  $t_i$  and  $s_i$ , so that once  $L_i$  and  $U_i$  are given, we let  $t_{i+1}$  be the minimum choice so that the resulting  $Q$ -value is  $\geq 1$ . More precisely, we let

$$(5.1) \quad t_{i+1} = \min \left\{ \ell : \frac{(1 + \omega)^b}{(1 + Q(U_i))^\ell (1 + \omega Q(L_i))^{b - \ell}} \geq 1 \right\}.$$

And similarly, we let

$$(5.2) \quad s_{i+1} = \max \left\{ \ell : \frac{(1 + \omega)^b}{(1 + Q(U_i))^\ell (1 + Q(L_i))^{b - \ell}} \leq 1 \right\}.$$

The recursion starts with  $U_1$  being the graph of a single node and  $L_1$  being the empty set, so that  $Q(U_1) = \lambda/\omega$  and  $Q(L_1) = 0$ . Observe that, by definition,  $s_{i+1} \in \{t_{i+1}, t_{i+1} + 1\}$  and that the construction guarantees that the values  $Q(L_i)$  are at most 1, and the values  $Q(U_i)$  are at least 1. The following simple lemma justifies the correctness of our construction.

LEMMA 5.1.

$$\lim_{i \rightarrow \infty} Q(U_i)/Q(L_i) = 1.$$

*Proof.* It is easy to see that either  $t_i = s_i$  (meaning that  $Q(L_i) = Q(U_i) = 1$ ), or  $t_i = s_i - 1$ , which implies that

$$\frac{Q(U_i)}{Q(L_i)} = \frac{1 + \omega Q(U_{i-1})}{1 + \omega Q(L_{i-1})} < \frac{Q(U_{i-1})}{Q(L_{i-1})}.$$

Therefore the ratio is shrinking and bounded from below by 1. Suppose the limit is not 1 but some value  $q > 1$ , which implies that  $Q(U_i)/Q(L_i) > q$  for all  $i$ . Then we have the following:

$$\begin{aligned}
 \frac{Q(U_{i-1})}{Q(L_{i-1})} - \frac{Q(U_i)}{Q(L_i)} &= \frac{Q(U_{i-1})}{Q(L_{i-1})} - \frac{1 + \omega Q(U_{i-1})}{1 + \omega Q(L_{i-1})} \\
 &= \frac{Q(U_{i-1}) - Q(L_{i-1})}{(1 + \omega Q(L_{i-1}))Q(L_{i-1})} \\
 &\geq \frac{(q - 1)Q(L_{i-1})}{Q(L_{i-1})(1 + \omega)} && \text{since } Q(U_i)/Q(L_i) > q \\
 &= \frac{q - 1}{1 + \omega} \\
 &> 0 && \text{since } q > 1.
 \end{aligned}$$

Therefore as long as  $q > 1$ , we show that the difference between the ratios for each step  $i$  is at least some positive constant, which is impossible. Hence the assumption is false, and it must be the case that  $q = 1$ .  $\square$

By this lemma, it is easy to check that if we let  $T_i^*$  be equal to either  $U_i$  or  $L_i$ , then  $Q(T_i^*) \rightarrow 1$ . Indeed, we can show that the additive error decreases exponentially fast. The following lemma indicates that this is the case for  $\omega < 1$  (although a similar result holds for any  $\omega$ ).

LEMMA 5.2. *Let  $\epsilon_i^+$  be the value of  $Q(U_i) - 1$  and let  $\epsilon_i^-$  be the value of  $1 - Q(L_i)$ ; then*

$$\epsilon_{i+1}^+ + \epsilon_{i+1}^- \leq \omega(\epsilon_i^+ + \epsilon_i^-).$$

*Proof.* Note that by algebraic manipulations, we have

$$(5.3) \quad \frac{(1 + \omega)^b}{(1 + \omega Q(U_i))^j (1 + \omega Q(L_i))^{b-j}} = \frac{1}{(1 + \frac{\omega}{1+\omega} \epsilon_i^+)^j (1 - \frac{\omega}{1+\omega} \epsilon_i^-)^{b-j}}.$$

Now, let  $k$  be the largest index  $j$  over  $[b]$  such that the denominator in the right-hand side of the previous expression is less than 1. Therefore,  $k + 1$  is the least index such that the denominator is greater than 1. Then, by applying (5.3) for  $Q(U_{i+1})$  and  $Q(L_{i+1})$ ,

$$\begin{aligned}
 \epsilon_{i+1}^+ + \epsilon_{i+1}^- &= \frac{1}{(1 + \frac{\omega}{1+\omega} \epsilon_i^+)^k (1 - \frac{\omega}{1+\omega} \epsilon_i^-)^{b-k}} - \frac{1}{(1 + \frac{\omega}{1+\omega} \epsilon_i^+)^{k+1} (1 - \frac{\omega}{1+\omega} \epsilon_i^-)^{b-k-1}} \\
 &= \frac{\frac{\omega}{1+\omega} (\epsilon_i^+ + \epsilon_i^-)}{(1 + \frac{\omega}{1+\omega} \epsilon_i^+)^{k+1} (1 - \frac{\omega}{1+\omega} \epsilon_i^-)^{b-k}} \\
 &\leq \frac{\frac{\omega}{1+\omega} (\epsilon_i^+ + \epsilon_i^-)}{1 - \frac{\omega}{1+\omega} \epsilon_i^-} \quad \text{by the above property of } k + 1 \\
 &\leq \omega(\epsilon_i^+ + \epsilon_i^-). \quad \square
 \end{aligned}$$

Coming back to the original tree-boundary notation, let  $\Gamma_h^1$  be the boundary corresponding to the trimming of the tree  $U_h$  and let  $\Gamma_h^2$  be the boundary corresponding to the trimming of the tree  $L_h$ . By our construction, for any vertex  $v$  on the tree of height  $h$ , the measure from  $\mu_{h, \Gamma_h^1}$  (or  $\mu_{h, \Gamma_h^2}$ ) projected onto the space of the independent sets of the subtree rooted at  $v$  with the boundary condition corresponding to the appropriate part of  $\Gamma$  and the parent of  $v$  being unoccupied is either  $\mu_{i, \Gamma_i^1}$  or  $\mu_{i, \Gamma_i^2}$ , where  $i$  is the distance of  $v$  away from the leaves on  $T_h$ . Conditioned on the parent of  $v$  being unoccupied in the broadcast process defined in the introduction, we would

occupy  $v$  with probability  $\omega/(1 + \omega)$ . Therefore, in the above construction, the probability that  $v$  is occupied (or rather unoccupied) is close to the desired probability, and the error will decay exponentially fast with the distance from the leaves. This is formally stated in the following corollary of Lemma 5.2.

**COROLLARY 5.3.** *Given any  $\omega < 1$  and the complete tree of height  $i$ , for  $\Gamma$  equal to  $\Gamma_i^1$  or  $\Gamma_i^2$  inductively constructed above, we have*

$$\left| \mu_{i,\Gamma}(\sigma(r) = 0) - \frac{1}{1 + \omega} \right| \leq \omega^{i-1} \lambda/b.$$

Throughout the rest of this section it is assumed that we are dealing with the boundary conditions  $\{\Gamma_h^1\}_{h>1}$  and  $\{\Gamma_h^2\}_{h>1}$  constructed above. We will then show that for every  $\omega = (1 + \delta) \ln b/b$ , whenever  $\delta > 0$ , under these two boundary conditions the Glauber dynamics on the hard-core model slows down. As we know from Corollary 5.3, the error of the marginal goes down very fast, so that roughly we can think of the marginal distribution of the configurations on the tree from the root to the vertices a few levels above the leaves as being close to the broadcasting measure. In fact, by following the same proof outline as we did in section 4, we are able to prove the same lower bound in the hard-core model for these boundaries. To do that we need a slight generalization of the reconstruction algorithm and extensions of the corresponding lemmas used in that section to handle the errors in the marginal probabilities.

To generalize the notion of a reconstruction algorithm to the case of a boundary condition, we need to add an extra parameter  $\ell$  depending only on  $\omega$  and  $b$ . We will essentially ignore the bottom  $\ell$  levels in the analysis and we will use that, for the top  $h - \ell$  levels, the marginal probabilities are close to those of the broadcasting tree. We define a *reconstruction algorithm with a parameter  $\ell$*  for the tree  $T_h$  with boundary condition  $\Gamma$  as a function  $A_\ell : \Omega(L_{h-\ell}) \rightarrow \{0, 1\}$ . The algorithm  $A_\ell$  takes the configurations of the vertices at height  $h - \ell$  as the input and tries to compute the configuration at the root. For any  $\sigma \in \Omega(T_{h,\Gamma})$ , the sensitivity is defined as

$$S_{\ell,A}(\sigma) = \frac{1}{n} \# \{v \in L_{h-\ell} : A_\ell(\sigma_{h-\ell}^v) \neq A_\ell(\sigma_{h-\ell})\}.$$

The average sensitivity of the algorithm at height  $h - \ell$  with respect to the boundary  $\Gamma$  is defined as

$$\bar{S}_{\ell,A}^\Gamma = \mathbf{E}_\sigma [S_{\ell,A}(\sigma) \mathbf{1}(A_\ell(\sigma_{h-\ell}) = 1)].$$

And the effectiveness is defined as

$$r_{\ell,A}^\Gamma = \min_{x \in \{0,1\}} [\mu_{h,\Gamma}(A_\ell(\sigma_{h-\ell}) = x \text{ and } \sigma(r) = x) - \mu_{h,\Gamma}(A_\ell(\sigma_{h-\ell}) = x) \mu_{h,\Gamma}(\sigma(r) = x)].$$

We can show the analogue of Theorem 3.3 in this setting.

**THEOREM 5.4.** *Suppose that  $A_\ell$  is an effective reconstruction algorithm. Then, it is the case that the spectral gap  $c_{gap}$  of the Glauber dynamics for the hard-core model on the tree of height  $h$  with boundary condition  $\Gamma$  satisfies  $c_{gap} = O(\bar{S}_{\ell,A}^\Gamma)$ , and hence the relaxation time of this Glauber dynamics satisfies  $T_{relax} = \Omega(1/\bar{S}_{\ell,A}^\Gamma)$ .*

To bound the average sensitivity for the boundary conditions  $\Gamma_h^1$  and  $\Gamma_h^2$  constructed above, we again use the same BW algorithm we analyzed for the broadcasting tree. As in (4.1) and (4.2), it is again enough to bound the probability

$$(5.4) \quad \mu_{h,\Gamma_h}(\text{BW}_\ell(\sigma_{h-\ell}) = 1 \text{ and } \text{BW}_\ell(\sigma_{h-\ell}^{z^*}) = 0)$$

for a fixed vertex  $z^*$  at a distance  $\ell$  from the leaves. However, in this case, this probability will not be the same for all  $z^*$ . Let the path  $\mathcal{P}$  from  $z^*$  to the root  $r$  be  $u_\ell = z^*, u_{\ell+1}, u_{\ell+2}, \dots, u_h = r$ .

As in the proof of Lemma 4.1 in section 4, let  $\hat{N}(u_i) = N(u_i) \setminus \{u_{i-1}\}$  denote the children of  $u_i$  different from  $u_{i-1}$ . For  $i > \ell$ , consider some  $w \in \hat{N}(u_i)$ . Let  $\Gamma(w)$  be the boundary condition  $\Gamma_h$  restricted to the subtree  $T_w$  of  $T_h$  rooted at the vertex  $w$ . These subtrees are of height  $i$ . Note that by our construction of the boundary conditions,  $\Gamma(w) = \Gamma_{i-1}^1$  or  $\Gamma(w) = \Gamma_{i-1}^2$ . Then (5.4) can be calculated by the following lemma, which is the analogue of Lemma 4.1 for the broadcasting tree.

LEMMA 5.5.

$$\begin{aligned} \mu_{h,\Gamma_h}(BW_\ell(\sigma_{h-\ell}) = 1 \text{ and } BW_\ell(\sigma_{h-\ell}^{z^*}) = 0) \\ \leq \mathbf{E}_\sigma \left[ \prod_{i>\ell:\sigma(u_i)=0} \prod_{w \in \hat{N}(u_i)} \mu_{i-1,\Gamma(w)}(\eta : BW_\ell(\eta) = 0) \right], \end{aligned}$$

where the expectation is over the measure  $\mu_{h,\Gamma_h}$ , and for each  $i$  and  $w \in \hat{N}(u_i)$ , the configuration  $\eta$  is a random configuration on the subtree rooted at  $w$  chosen from the probability measure  $\mu_{i-1,\Gamma(w)}$ .

*Proof.* Let  $BW_\ell(\sigma_{h-\ell}, w)$  denote the labeling of the algorithm on vertex  $w$  from the input configurations  $\sigma$  on the vertices at height  $h - \ell$  (i.e., level  $\ell$ ). By a similar argument as in the proof of Lemma 4.1,

$$\begin{aligned} \mu_{h,\Gamma_h}(BW_\ell(\sigma_{h-\ell}) = 1 \text{ and } BW_\ell(\sigma_{h-\ell}^{z^*}) = 0) \\ \leq \mu_{h,\Gamma_h}(\sigma : \forall i > \ell, w \in \hat{N}(u_i), BW_\ell(\sigma_{h-\ell}, w)) \\ \leq \sum_{\eta \in \{0,1\}^{|\mathcal{P}|}} \mu_{h,\Gamma_h}(\sigma : \sigma_{\mathcal{P}} = \eta) \prod_{i>\ell} \prod_{w \in \hat{N}(u_i)} \mathbf{Pr}_\sigma [BW_\ell(\sigma_{h-\ell}, w) = 0 \mid \sigma(u_i) = \eta(u_i)] \\ \leq \sum_{\eta \in \{0,1\}^{|\mathcal{P}|}} \mu_{h,\Gamma_h}(\sigma : \sigma_{\mathcal{P}} = \eta) \prod_{i>\ell:\eta(u_i)=0} \prod_{w \in \hat{N}(u_i)} \mu_{i-1,\Gamma(w)}(\eta : BW_\ell(\eta) = 0) \\ = \mathbf{E}_\sigma \left[ \prod_{i>\ell:\sigma(u_i)=0} \prod_{w \in \hat{N}(u_i)} \mu_{i-1,\Gamma(w)}(\eta : BW_\ell(\eta) = 0) \right]. \quad \square \end{aligned}$$

To bound  $\mu_{i-1,\Gamma(w)}(\eta : BW_\ell(\eta) = 0)$  for every  $i > \ell$  and  $w \in \hat{N}(u_i)$ , we proceed along the lines of Lemma 4.2, but extra care is required to deal with the errors in the marginal probabilities which were bounded in Corollary 5.3. Here and throughout the remainder of the paper, we define  $\ell_0 = \ell(\lambda, b)$  to be

$$(5.5) \quad \ell_0 = \min \left\{ \ell : \left| \frac{\mu_{i,\Gamma_i^1}(\eta : \eta(r) = 0)}{1/(1+\omega)} - 1 \right| \leq \left( \exp \left( \frac{1.01\omega^2 b}{\lambda} \right) - 1 \right) \forall i > \ell \right\}.$$

The existence of such a constant  $\ell(\lambda, b)$  is guaranteed by Lemma 5.1. Moreover, from Corollary 5.3 we can deduce an explicit value for  $\ell_0$ , provided that  $\omega < 1$ . For every  $i \geq \ell$ , let  $f_{i,1} = \mu_{i,\Gamma_i^1}(\eta : BW_\ell(\eta) = 0)$ , and similarly let  $f_{i,2} = \mu_{i,\Gamma_i^2}(\eta : BW_\ell(\eta) = 0)$ . We will use the following lemma to bound  $f_{i,1}$  and  $f_{i,2}$ .

LEMMA 5.6. *For all  $\delta > 0$ , all  $b \geq b_0(\delta)$ , there exist  $\ell_0 = \ell(\lambda, b)$  such that for all  $i > \ell_0$ , the following bounds hold:*

$$f_{i,1} \leq \frac{1.01^{1/b}}{1+\omega} \quad \text{and} \quad f_{i,2} \leq \frac{1.01^{1/b}}{1+\omega}.$$

*Proof.* Similar to the proof of Lemma 4.2, the proof is again by induction. Here we take  $\ell = \ell_0$ . Let  $\bar{t}_i = b - t_i$  and  $\bar{s}_i = b - s_i$  for simplicity. Recall that we define  $t_i$  and  $s_i$  in (5.1) and (5.2). Again, we can derive the recurrences in exactly the same way as in Lemma 4.2.

For the base case  $i = \ell$ , by the definition the algorithm will label the vertices on level  $\ell$  to be the same as their actual configurations. For instance, for the boundary condition  $\Gamma_\ell^1$  on the complete tree of height  $\ell$ , the root is unoccupied with probability  $\mu_{\ell, \Gamma_\ell^1}(\eta(r) = 0)$  for a random configuration  $\eta$ . Therefore,

$$f_{\ell,1} = \mu_{\ell, \Gamma_\ell^1}(\eta(r) = 0), \quad f_{\ell,2} = \mu_{\ell, \Gamma_\ell^2}(\eta(r) = 0).$$

For the case  $i = \ell + 1$ , for a random configuration  $\eta \sim \mu_{\ell+1, \Gamma_{\ell+1}^1}$ , in order for the root  $r$  at level  $\ell + 1$  to be labeled as 0 (unoccupied) in the algorithm, at least one child of  $r$  should be occupied in  $\eta$  since the algorithm takes input at level  $\ell$  by the definition. This requires us to unoccupy the root with probability  $\mu_{\ell+1, \Gamma_{\ell+1}^1}(\eta(r) = 0)$  and then have at least one child occupied in  $\eta$ , which happens with probability  $(1 - f_{\ell,1}^{t_{\ell+1}} f_{\ell,2}^{\bar{t}_{\ell+1}})$ . Note that the boundary condition for the subtree rooted at each child is not the same. There are  $t_{\ell+1}$  trees with boundary condition  $\Gamma_\ell^1$  and  $\bar{t}_{\ell+1}$  trees with boundary condition  $\Gamma_\ell^2$  by the definition of  $\Gamma_{\ell+1}^1$ . The same argument holds for  $f_{\ell+1,2}$ . Therefore,

$$f_{\ell+1,1} = \mu_{\ell+1, \Gamma_{\ell+1}^1}(\eta(r) = 0) \left(1 - f_{\ell,1}^{t_{\ell+1}} f_{\ell,2}^{\bar{t}_{\ell+1}}\right),$$

$$f_{\ell+1,2} = \mu_{\ell+1, \Gamma_{\ell+1}^2}(\eta(r) = 0) \left(1 - f_{\ell,1}^{s_{\ell+1}} f_{\ell,2}^{\bar{s}_{\ell+1}}\right).$$

And, for each  $i > \ell$ , by the same argument as in Lemma 4.2 and taking the boundary conditions into consideration as we did for  $f_{\ell+1,1}$  and  $f_{\ell+1,2}$ , we have

$$(5.6) \quad f_{i+1,1} = \mu_{i, \Gamma_i^1}(\eta(r) = 1) \left(1 - \left(1 - f_{i-1,1}^{t_i} f_{i-1,2}^{\bar{t}_i}\right)^{t_{i+1}} \left(1 - f_{i-1,1}^{s_i} f_{i-1,2}^{\bar{s}_i}\right)^{\bar{t}_{i+1}}\right) + \mu_{i, \Gamma_i^1}(\eta(r) = 0) \left(1 - f_{i,1}^{t_{i+1}} f_{i,2}^{\bar{t}_{i+1}}\right),$$

$$(5.7) \quad f_{i+1,2} = \mu_{i, \Gamma_i^2}(\eta(r) = 1) \left(1 - \left(1 - f_{i-1,1}^{t_i} f_{i-1,2}^{\bar{t}_i}\right)^{s_{i+1}} \left(1 - f_{i-1,1}^{s_i} f_{i-1,2}^{\bar{s}_i}\right)^{\bar{s}_{i+1}}\right) + \mu_{i, \Gamma_i^2}(\eta(r) = 0) \left(1 - f_{i,1}^{s_{i+1}} f_{i,2}^{\bar{s}_{i+1}}\right).$$

Our goal now is to show by induction that  $f_{i,1}, f_{i,2} \leq \frac{1.01^{1/b}}{1+\omega}$  for all  $i \geq \ell = \ell_0$ . From the definition of  $\ell_0$  in (5.5), the base case is simple:

$$f_{\ell+1,1} \leq f_{\ell,1} \leq \mu_{\ell, \Gamma_\ell^1}(\eta(r) = 0) \leq \frac{1}{1+\omega} \exp\left(\frac{1.01\omega^2 b}{\lambda}\right),$$

and the last term is less than or equal to  $\frac{1.01^{1/b}}{1+\omega}$  for  $b \geq b_0(\delta)$ . Similarly, it is the case that  $f_{\ell+1,2} \leq f_{\ell,2} \leq \frac{1.01^{1/b}}{1+\omega}$ . Assuming the inductive hypothesis, by algebraic calculations, we can get from the above recurrence (5.7) that

$$(f_{i+1,1})^b \leq \left[ \frac{\omega}{1+\omega} \left(1 - \left(1 - \frac{1.01\omega}{\lambda}\right)^b\right) + \frac{1}{1+\omega} \right]^b \exp\left(\frac{1.01(\omega b)^2}{\lambda}\right)$$

$$\leq \frac{\exp(1.01(\omega b)^2/\lambda)}{(1+\omega)^b} \exp\left(\frac{1.01(\omega b)^2}{\lambda}\right) \quad \text{as in the proof of Lemma 4.2}$$

$$\begin{aligned}
 &= \frac{\exp(2(1.01)(\omega b)^2/\lambda)}{(1 + \omega)^b} \\
 &\leq \frac{1.01}{(1 + \omega)^b} \quad \text{for } b \geq b_0(\delta), \text{ by the definition of } b_0(\delta) \text{ in (4.4)}.
 \end{aligned}$$

This proves  $f_{i+1,1} \leq \frac{1.01^{1/b}}{1+\omega}$  by induction, and a similar proof can be done for  $f_{i+1,2}$ .  $\square$

It is also not hard to show that the BW algorithm under the same setting as in Lemma 5.6 is effective.

**PROPOSITION 5.7.** *For all  $\delta > 0$  and  $b > b_0(\delta)$ , the BW algorithm is an effective reconstruction algorithm to recover the configuration at the root from the configurations at distance  $\ell(\lambda, b)$  from the leaves.*

*Proof.* We use the same notation as in the proof of Lemma 5.6. Let  $\bar{t}_i = b - t_i$  and  $\bar{s}_i = b - s_i$ , where  $t_i$  and  $s_i$  are defined in (5.1) and (5.2). Then,

$$\begin{aligned}
 \Pr_{\sigma \sim \mu_{h, \Gamma_h^1}} [\text{BW}(\sigma) = 0 \mid \sigma(r) = 0] &= 1 - \left[ (f_{h-1,1})^{t_h} (f_{h-1,2})^{\bar{t}_h} \right], \\
 \Pr_{\sigma \sim \mu_{h, \Gamma_h^1}} [\text{BW}(\sigma) = 1 \mid \sigma(r) = 1] &= \left[ 1 - (f_{h-2,1})^{t_{h-1}} (f_{h-2,2})^{\bar{t}_{h-1}} \right]^{t_h} \\
 &\quad \times \left[ 1 - (f_{h-2,1})^{s_{h-1}} (f_{h-2,2})^{\bar{s}_{h-1}} \right]^{\bar{t}_h}.
 \end{aligned}$$

These recursions follow easily by noticing that  $\text{BW}(\sigma) = 0$  iff it is not true that  $\text{BW}(\sigma_i) = 0$  for all  $i = 1, \dots, b$ , where  $\sigma_i$  is the restriction of  $\sigma$  to the tree subtended at the  $i$ th children of the root, and also that  $\text{BW}(\sigma) = 1$  iff it is not true that  $\text{BW}(\sigma_i) = 0$  for all  $i = 1, \dots, b^2$ , where  $\sigma_i$  is the restriction of  $\sigma$  to the tree subtended at the  $i$ th grandchildren of the root. Now, from these recurrences and the bounds stated in Lemma 5.6, we deduce that

$$\begin{aligned}
 &\mu_{h, \Gamma_h^1}(\text{BW}(\sigma) = 0, \sigma(r) = 0) - \mu_{h, \Gamma_h^1}(\text{BW}(\sigma) = 0)\mu_{h, \Gamma_h^1}(\sigma(r) = 0) \\
 &= \Omega \left( 1 - \frac{1.01}{(1 + \omega)^b} - \frac{1.01^{1/b}}{1 + \omega} \right).
 \end{aligned}$$

Now, notice that

$$1 - \frac{1.01^{1/b}}{1 + \omega} \geq \frac{1}{b} \left[ \frac{(1 + \delta) \ln b - 0.01}{(1 + \omega)} \right].$$

Also, for  $b \geq b_0(\delta)$  as defined in (4.4), we have that

$$\frac{1.01}{(1 + \omega)^b} \leq \frac{1}{b} \left[ \frac{0.01}{2(1 + \delta) \ln b} \right].$$

Therefore, effectiveness, with rate roughly  $\frac{(1+\delta)\ln b}{b}$ , holds for all  $b \geq b_0(\delta)$ . The same result holds for  $\mu_{h, \Gamma_h^2}(\cdot)$ .  $\square$

Then, we are able to again bound  $\bar{S}_{\ell, \text{BW}}^\Gamma$  for  $\Gamma = \Gamma_h^1$  or  $\Gamma_h^2$ , proving the following theorem, which completes the proof of part 2(b) in Theorem 1.1.

**THEOREM 5.8.** *Let  $\delta > 0$ , and let  $\omega = (1 + \delta) \ln b/b$ . For all  $b \geq b_0(\delta)$ , it is the case that*

$$T_{\text{relax}} = \Omega(n^d), \quad \text{where } d = \left( 1 + \frac{\ln(\lambda/(1.01\omega b)^2)}{2 \ln b} \right).$$

*Proof.* We take  $\ell$  as  $\ell_0 = \ell(\lambda, b)$ , as in Lemma 5.6. Now, due to Lemma 5.5, we have that

$$\bar{S}_{\ell, \text{BW}}^\Gamma = O\left(\frac{1}{n} \sum_{z^* \in \text{level } \ell} \mathbf{E}_{\sigma \sim \mu_{h, \Gamma_h}} \left[ \left( \frac{1.01\omega(1+\omega)}{\lambda} \right)^{\#\{i: \sigma(u_i)=0\}} \right]\right).$$

The following lemma bounds the expectation and the proof is presented in the next section.

LEMMA 5.9. *For all  $\delta > 0$ , all  $b > b_0(\delta)$ , setting  $\omega = (1 + \delta) \ln b/b$ , we have that for any leaf  $z^*$  in level  $\ell$  and the corresponding path  $\mathcal{P}$  from the root to  $z^*$ ,*

$$\mathbf{E}_{\sigma \sim \mu_{h, \Gamma_h}} \left[ \left( \frac{1.01\omega(1+\omega)}{\lambda} \right)^{\#\{i: \sigma(u_i)=0\}} \right] = O\left(\left(1.01 \frac{\omega}{\lambda^{1/2}}\right)^h\right).$$

Since  $\ell_0$  is a constant independent of  $n$ , just as the argument at the end of section 4, we can deduce that for  $\delta > 0$ ,  $b > b_0(\delta)$ , setting  $\omega = (1 + \delta) \ln b/b$ ,

$$\bar{S}_{\text{BW}} = O\left(\left[\frac{1.01\omega}{\lambda^{1/2}}\right]^h\right) = O\left(n^{-\left[1 + \frac{\ln(\lambda/(1.01\omega b)^2)}{2 \ln b}\right]}\right).$$

Now, from Proposition 5.7, the BW algorithm is effective for  $\omega > (1 + \delta) \ln b/b$ ,  $b > b_0(\delta)$ . Therefore, Theorem 5.4 applies, and the conclusion follows (trivially for  $b < b_0(\delta)$ ).  $\square$

**6. Technical lemmas.** Here we provide all the technical lemmas that are needed in the previous sections for calculating the expectations. Recall that in the broadcasting model, we fix a leaf  $z^*$  and take sample  $\sigma$  from distribution  $\nu_h$ . We need to calculate the following expectation:

$$\mathbf{E}_{\sigma \sim \nu_h} \left[ \left( \frac{1.01\omega(1+\omega)}{\lambda} \right)^{\#\{i: \sigma(u_i)=0\}} \right],$$

where  $u_i$  are vertices on the path  $\mathcal{P}$  from  $z^*$  to the root. Notice that here we let  $u_0$  be the root and  $u_h$  be the leaf  $z^*$ , which reverses the order we used for  $u_i$  in previous sections. Observe that the random configurations for each  $u_i$  are essentially Markovian with respect to  $i$  due to the spatial Markov property of hard-core model. Therefore, we will first prove the following results concerning a class of Markov chains and then apply it to calculate the expectations for Lemmas 4.3 and 5.9.

LEMMA 6.1. *Let  $\zeta_0, \zeta_1, \dots$  be a Markov process with state space  $\{0, 1\}$  such that  $\zeta_0 = 0$  and with transition rates  $p_{0 \rightarrow 0} = p$ ,  $p_{0 \rightarrow 1} = q$ ,  $p_{1 \rightarrow 0} = 1$ ,  $p_{1 \rightarrow 1} = 0$ . Let  $N_h = \#\{1 \leq i \leq h : \zeta_i = 0\}$ .*

1. *Then, for any  $a > 0$ ,*

$$\mathbf{E}[a^{N_h}] = O\left(\left(\frac{pa}{2} \left[1 + \sqrt{1 + 4q/(ap^2)}\right]\right)^h\right).$$

2. *Moreover, if  $\bar{\zeta}_0, \bar{\zeta}_1, \dots$  is an inhomogeneous chain with transition rates  $p_{0 \rightarrow 0}^i = p_i$ ,  $p_{0 \rightarrow 1}^i = q_i$ ,  $p_{1 \rightarrow 0}^i = 1$ ,  $p_{1 \rightarrow 1}^i = 0$  and such that for some  $\gamma > 0$ ,  $|\frac{p_{0 \rightarrow 0}^i}{p} - 1|$ . Then, if  $\bar{N}_h = \#\{1 \leq i \leq h : \bar{\zeta}_i = 0\}$ , we have that for any  $a > 0$*

$$\mathbf{E}[a^{\bar{N}_h}] \leq (1 + \gamma) \mathbf{E}[(a(1 + \gamma))^{N_h}].$$

*Proof.* Let  $\tau_1 = \min\{\ell : \zeta_\ell = 1\}$ , and for  $i \geq 1$ , let  $\tau_{i+1} = \min\{\ell - \tau_i : \ell \geq \tau_i \text{ and } \zeta_\ell = 1\}$ . So,  $\tau_1$  is the index of the first occurrence of state 1 and  $\tau_2, \tau_3, \dots$  are the distance between subsequent occurrences of state 1 in the sequence. Also, let  $\tilde{\tau} = \min\{h - \ell : \ell \leq h \text{ and } \tau_\ell = 1\}$ , that is, the distance between  $h$  and the last occurrence of state 1 in the sequence  $\zeta_0, \zeta_1, \dots, \zeta_h$ . It is easy to see that

$$\begin{aligned} \Pr [N_h = h - k, \tau_1 = t_1, \dots, \tau_k = t_k, \tilde{\tau} = \tilde{t}] \\ = \begin{cases} p^{h-2k} q^k & \text{if } \tilde{t} \geq 1, 0 \leq k \leq \lfloor h/2 \rfloor, \\ p^{h-2k+1} q^k & \text{if } \tilde{t} = 0, 1 \leq k \leq \lfloor (h+1)/2 \rfloor. \end{cases} \end{aligned}$$

Thus, adding up over all the possible choices of  $t_1, \dots, t_k, \tilde{t}$ , having in mind the restrictions  $t_1 \geq 1, t_2 \geq 2, \dots, t_k \geq 2$ , and  $t_1 + \dots + t_k + \tilde{t} = h$ , we obtain

$$\begin{aligned} \Pr [N_h = h - k \text{ and } \zeta_h = 0] &= \binom{h-k}{k} p^{h-2k} q^k && \text{for } 0 \leq k \leq \lfloor h/2 \rfloor, \\ \Pr [N_h = h - k \text{ and } \zeta_h = 1] &= \binom{h-k}{k-1} p^{h-2k+1} q^k && \text{for } 1 \leq k \leq \lfloor (h+1)/2 \rfloor; \end{aligned}$$

therefore

$$(6.1) \quad \mathbf{E} [a^{N_h}] = \sum_{k=0}^{\lfloor h/2 \rfloor} \binom{h-k}{k} p^{h-2k} q^k a^{h-k} + \sum_{k=1}^{\lfloor (h+1)/2 \rfloor} \binom{h-k}{k-1} p^{h-2k+1} q^k a^{h-k}.$$

Now, for the first term, we have that

$$\sum_{k=0}^{\lfloor h/2 \rfloor} \binom{h-k}{k} p^{h-2k} q^k a^{h-k} = (pa)^h \sum_{k=0}^{\lfloor h/2 \rfloor} \binom{h-k}{k} x^k,$$

where  $x = \frac{q}{ap^2}$ . By the standard saddle point formula, after noticing that the function

$$\phi(t) = \lim_{h \rightarrow \infty} h^{-1} \ln \left[ \binom{h-th}{th} x^{th} \right] = (1-t) \mathbf{H} \left( \frac{t}{1-t} \right) + t \ln(x)$$

(where  $\mathbf{H}$  stands for natural entropy) reaches its maximum at  $t^* = \frac{1}{2}(1 - \epsilon)$ , where  $\epsilon = 1/\sqrt{1+4x}$  and  $\phi''(t^*) = \frac{-4}{\epsilon(1-\epsilon)(1+\epsilon)}$ , we have that

$$(6.2) \quad \begin{aligned} \sum_{k=0}^{\lfloor h/2 \rfloor} \binom{h-k}{k} x^k &= (1 + o_h(1)) \sqrt{\frac{(1-t^*)}{t^*(1-2t^*)|\phi''(t^*)|}} \times e^{h\phi(t^*)} \\ &= \frac{(1+\epsilon)}{2} \left( \frac{1+\sqrt{1+4x}}{2} \right)^h. \end{aligned}$$

For the second term in (6.1), we have that

$$\sum_{k=1}^{\lfloor (h+1)/2 \rfloor} \binom{h-k}{k-1} p^{h-2k+1} q^k a^{h-k} = p(pa)^h \sum_{k=1}^{\lfloor (h+1)/2 \rfloor} \binom{h-k}{k-1} x^k,$$

Using a similar saddle point estimate, we have that

$$(6.3) \quad \sum_{k=1}^{\lfloor (h+1)/2 \rfloor} \binom{h-k}{k-1} x^k = (1 + o_h(1)) \frac{(1-\epsilon^2)}{4\epsilon} \left( \frac{1+\sqrt{1+4x}}{2} \right)^h.$$

Now, combining the asymptotics (6.2) and (6.3) into (6.1), part 1 follows.

For part 2, using the same notation as above, we have that

$$\begin{aligned} & \Pr [\bar{N}_h = h - k, \tau_1 = t_1, \dots, \tau_k = t_k, \tilde{\tau} = \tilde{t}] \\ & \leq \begin{cases} (1 + \gamma)^{h-k} p^{h-2k} q^k & \text{if } \tilde{t} \geq 1, 0 \leq k \leq \lfloor h/2 \rfloor, \\ (1 + \gamma)^{h-k+1} p^{h-2k+1} q^k & \text{if } \tilde{t} = 0, 1 \leq k \leq \lfloor (h + 1)/2 \rfloor. \end{cases} \end{aligned}$$

Therefore,

$$\begin{aligned} \Pr [\bar{N}_h = h - k \text{ and } \zeta_h = 0] & \leq \binom{h-k}{k} (1 + \gamma)^{h-k} p^{h-2k} q^k \quad \text{for } 0 \leq k \leq \lfloor h/2 \rfloor, \\ \Pr [\bar{N}_h = h - k \text{ and } \zeta_h = 1] & \leq \binom{h-k}{k-1} (1 + \gamma)^{h-k+1} p^{h-2k+1} q^k \quad \text{for } 1 \leq k \leq \lfloor (h + 1)/2 \rfloor. \end{aligned}$$

This leads to

$$\begin{aligned} \mathbf{E}[a^{\bar{N}_h}] & \leq \sum_{k=0}^{\lfloor h/2 \rfloor} \binom{h-k}{k} (1 + \gamma)^{h-k+1} p^{h-2k} q^k a^{h-k} \\ & \quad + \sum_{k=1}^{\lfloor (h+1)/2 \rfloor} \binom{h-k}{k-1} (1 + \gamma)^{h-k+1} p^{h-2k+1} q^k a^{h-k} \\ & = (1 + \gamma) \mathbf{E}[(a(1 + \gamma))^{N_h}]. \quad \square \end{aligned}$$

*Proof of Lemma 4.3.* Notice that, for  $\sigma \sim \nu_h$ ,  $\zeta_i := \sigma(u_i)$  is a Markov chain with state space  $\{0, 1\}$  and transition probabilities  $p_{0 \rightarrow 0} = 1/(1 + \omega)$  and  $p_{1 \rightarrow 0} = 1$ . To estimate  $\mathbf{E}_{\sigma \sim \nu_h} [\theta^{\#\{i: \zeta_i = 0\}}]$  for any  $\theta > 0$ , we apply the technical result in part 1 of Lemma 6.1. In fact, recalling the random variable  $N_h$  defined in Lemma 6.1, we have that

$$\begin{aligned} \mathbf{E}_{\sigma \sim \nu_h} [\theta^{\#\{i: \zeta_i = 0\}}] & = O(\mathbf{E}_{\sigma \sim \nu_h} [\theta^{\#\{i: \zeta_i = 0\}} : \zeta_0 = 0]) \\ & = O(\mathbf{E}[\theta^{N_h}]). \end{aligned}$$

Therefore, plugging in the asymptotic from the lemma for  $\theta = \frac{1.01\omega(1+\omega)}{\lambda}$ , we get

$$\mathbf{E}[\theta^{N_h}] = O\left(\left(\frac{1.01\omega}{2\lambda} \left[1 + \sqrt{1 + \frac{4\lambda}{1.01}}\right]\right)^h\right) \leq O\left(\left(\frac{1.01\omega}{\lambda^{1/2}}\right)^h\right).$$

For the last inequality we used the fact that  $1 + \sqrt{1 + 4\lambda/1.01} \leq 2\lambda^{1/2}$ , which holds for  $\lambda > (101)^2$ , and in particular, when  $\omega = (1 + \delta) \ln(b)/b$  and  $b > b_0(\delta)$ , where  $b_0(\delta)$  was defined in (4.4).  $\square$

*Proof of Lemma 5.9.* The proof goes along the lines of Lemma 4.3. For  $\sigma \sim \mu_{h, \Gamma_h}$ ,  $\zeta_i := \sigma(u_i)$  is an inhomogeneous Markov chain with state space  $\{0, 1\}$  and transition probabilities, for  $i \leq h - \ell(\lambda, b)$ , such that  $|\frac{p_{0 \rightarrow 0}}{1/(1+\omega)} - 1| \leq (\exp(\frac{1.01\omega^2 b}{\lambda}) - 1)$  (from (5.5)) and  $p_{1 \rightarrow 0} = 1$ . Now, to estimate  $\mathbf{E}_{\sigma \sim \nu_h} [\theta^{\#\{i: \zeta_i = 0\}}]$  for  $\theta > 0$ , we apply part 2 of Lemma 6.1. This time, recalling the random variables  $N_h$  and  $\bar{N}_h$  defined in such lemma, we have that

$$\begin{aligned} \mathbf{E}_{\sigma \sim \mu_{h, \Gamma_h}} [\theta^{\#\{i: \zeta_i = 0\}}] & = O\left(\frac{1}{1 + \omega} \mathbf{E}_{\sigma \sim \mu_{h, \Gamma_h}} [\theta^{\#\{i: \zeta_i = 0\}} : \zeta_0 = 0]\right) \\ & = O\left(\frac{1}{1 + \omega} \mathbf{E}_{\sigma \sim \mu_{h, \Gamma_h}} [\theta^{\#\{i \leq h - \ell: \zeta_i = 0\}} : \zeta_0 = 0]\right) \end{aligned}$$

$$\begin{aligned} &= O(\mathbf{E}[\theta^{\bar{N}_h - \epsilon}]) \\ &= O(\mathbf{E}[\theta^{\bar{N}_h}]) \\ &= O\left(\mathbf{E}\left[\left(\exp\left(\frac{1.01\omega^2 b}{\lambda}\right)\theta\right)^{\bar{N}_h}\right]\right). \end{aligned}$$

Now, plugging in the asymptotics for  $\theta = \frac{1.01\omega(1+\omega)}{\lambda}$ , we get

$$\mathbf{E}\left[\left(\exp\left(1.01(\omega b)^2/\lambda\right)\theta\right)^{\bar{N}_h}\right] = O\left(\left[\frac{1.01\omega}{2\lambda}\exp\left(1.01(\omega b)^2/\lambda\right)\Upsilon\right]^h\right),$$

where

$$\Upsilon = 1 + \sqrt{1 + \frac{4\lambda}{1.01}\exp\left(-1.01(\omega b)^2/\lambda\right)}.$$

Finally we use the inequality

$$\Upsilon \leq 2\lambda^{1/2}\exp\left(-\frac{1.01\omega^2 b}{\lambda}\right),$$

which holds whenever  $\omega = (1 + \delta)\ln b/b$  and  $b > b_0(\delta)$ .  $\square$

**7. Upper bounds of the relaxation time.** Before showing the main idea for our upper bound proofs, we first introduce some notation we use in this section. For a  $b$  dimensional vector  $\rho = (\rho_1, \dots, \rho_b)$ , where  $0 \leq \rho_i \leq 1$  for every  $1 \leq i \leq b$ , let  $\tau_\rho$  be the relaxation time of the following Glauber dynamics of the hard-core model on the star graph  $G^*$  with root  $r$  and  $b$  leaves  $\{w_1, \dots, w_b\}$ . The dynamics on the star graph  $G^*$  is defined as follows. Given an independent set  $X_t$ ,

1. choose a vertex  $v$  uniformly at random from  $\{r, w_1, \dots, w_b\}$ ;
2. if  $v = r$ , then set

$$X' = \begin{cases} X_t \cup \{v\} & \text{with probability } \lambda/(1 + \lambda), \\ X_t \setminus \{v\} & \text{with probability } 1/(1 + \lambda); \end{cases}$$

3. if  $v = w_i$  is a leaf of  $G^*$ , then set

$$X' = \begin{cases} X_t \cup \{w_i\} & \text{with probability } \rho_i, \\ X_t \setminus \{w_i\} & \text{with probability } 1 - \rho_i. \end{cases}$$

4. If  $X'$  is an independent set, then set  $X_{t+1} = X'$ ; otherwise set  $X_{t+1} = X_t$ .

Let  $\tau^* := \max_\rho\{\tau_\rho\}$  be defined as the worst-case relaxation time over all possible choices of  $\rho$ . Using the block dynamics approach of Martinelli [22], as used in [3, section 2.3] (see also [21] and [34] for similar results), it is not hard to show that the relaxation time of the above Glauber dynamics is exactly the same as that of the natural block dynamics which updates the configurations of a whole subtree of the root in one step, and hence the following lemma holds.

**LEMMA 7.1.** *For the complete tree of height  $H$  with any boundary condition on the leaves, the relaxation time  $T_{\text{relax}}$  of the Glauber dynamics of the hard-core model satisfies*

$$T_{\text{relax}} \leq (\tau^*)^H.$$

We omit the proof of the above lemma since it is essentially identical to that in [3, section 2.3].

Note that the relaxation time on the complete tree is quite sensitive to the boundary conditions. For example, as mentioned in the introduction, Martinelli, Sinclair, and Weitz [24] show that when the boundary condition is all even (or similarly for all odd), i.e., all the leaves are occupied when the height is even (respectively, odd) and all the leaves are unoccupied when the height is odd (even), then the mixing time is  $O(n \ln n)$  for all  $\lambda$ . In this paper we are considering all boundary conditions, and in our lower bound, we show there are boundary conditions that slow the Glauber dynamics. The lower bound on the relaxation time for the Glauber dynamics under those boundary conditions which we show that suffers the slowdown roughly matches up with the upper bound we prove here. The following lemma establishes such upper bound for  $\tau^*$ .

LEMMA 7.2. *For the Glauber dynamics of the hard-core model on  $G^*$ , the worst relaxation time over all the boundary conditions  $\rho$  satisfies*

$$\tau^* \leq 100(\lambda + 1)(b + 1) \ln^2(b + 1).$$

Therefore, by Lemmas 7.1 and 7.2, for any boundary condition on the leaves, the Glauber dynamics of the hard-core model on the complete tree of height  $H$  satisfy

$$T_{\text{relax}} \leq (\tau^*)^H \leq (100(\lambda + 1)(b + 1) \ln^2(b + 1))^{\log_b n} \leq n^d,$$

where

$$d = 1 + \frac{\ln(200(\lambda + 1) \ln^2(b + 1))}{\ln b}.$$

Now, if  $\omega \leq \frac{\ln b}{b}$ , we have that, for some constant  $c_0 > 0$ ,

$$d \leq 1 + \frac{c_0 \ln \ln b}{\ln b}.$$

On the other hand, for  $\delta > 0$  and  $\omega = (1 + \delta) \ln b / b$ , we instead get that for some constant  $c_1 > 0$ ,

$$d \leq 1 + \delta + \frac{c_1 \ln \ln b}{\ln b}.$$

This proves Theorem 1.1.

**7.1. Proof of Lemma 7.2.** We will analyze the following coupling  $\mathcal{L}$  of two copies  $(X_t), (Y_t)$  of the Glauber dynamics of the hard-core model on  $G^*$ . The coupling  $\mathcal{L}$  chooses the same random vertex  $v$  to update in both chains  $X_t$  and  $Y_t$ . If  $v = r$ , the root of  $G^*$ , and there is not an occupied leaf in either of the two copies, then  $r$  is coupled to be occupied with probability  $\lambda/(1 + \lambda)$  and unoccupied with probability  $1/(1 + \lambda)$  in both  $X_{t+1}$  and  $Y_{t+1}$ . If  $v = w_i$ , a leaf in  $G^*$ , and the root is unoccupied in both  $X_t$  and  $Y_t$ , then  $v$  is also coupled to be occupied or unoccupied in both  $X_{t+1}$  and  $Y_{t+1}$  with the corresponding probability. If in either  $X_t$  or  $Y_t$ , there is an occupied vertex among the neighbors of  $v$ , then each copy is updated independently with the corresponding probability.

Given a pair of configurations  $\eta, \eta' : G^* \rightarrow \{0, 1\}$ , we say that  $\eta \leq \eta'$  if  $\eta(r) \leq \eta'(r)$  and for every  $i = 1, \dots, b$ ,  $\eta(w_i) \geq \eta'(w_i)$ . Let  $\eta_{\max}$  and  $\eta_{\min}$  be the unique maximal

and minimal elements in this partial order, respectively. An important property of the coupling  $\mathcal{L}$  of the hard-core model in the star is *monotonicity*. Namely, if  $(X_t, Y_t)$  are such that  $X_t \preceq Y_t$ , then after applying one step of the coupled dynamics we have that  $X_{t+1} \preceq Y_{t+1}$ . More generally for bipartite graphs  $G$ , the hard-core model is a *monotone* system (see, e.g., [20, Chapter 22]) in the sense that if  $(X_t)$  and  $(Y_t)$  are two copies of the Glauber dynamics on the hard-core model on  $G$  and  $x_0 \preceq y_0$ , then there exists a one-step coupling  $\mathcal{C}$  of  $(X_t)$  and  $(Y_t)$  such that for all  $t \geq 0$ ,

$$\Pr_{\mathcal{C}} [X_t \preceq Y_t \mid X_0 = x_0, Y_0 = y_0] = 1.$$

In this case, we say  $(X_t)$  is stochastically dominated by  $(Y_t)$  and denote it as  $X_t \preceq^d Y_t$ .

Using monotonicity of the coupling  $\mathcal{L}$ , we have that

$$\Pr_{\mathcal{L}} [X_t \neq Y_t \mid X_0, Y_0] \leq \Pr_{\mathcal{L}} [X_t \neq Y_t \mid X_0 = \eta_{\max}, Y_0 = \eta_{\min}],$$

that is, the worst-case initial configurations for the coupling probability are the maximal and minimal configurations.

Therefore, using (2.1) and the coupling lemma, to prove Lemma 7.2 it is enough to show

(7.1)

$$\Pr_{\mathcal{L}} [X_T \neq Y_T \mid X_0 = \eta_{\max}, Y_0 = \eta_{\min}] \leq 1/2e \text{ for } T = 100(1 + \lambda)(b + 1) \ln^2(b + 1).$$

We will use the censoring technique of Peres and Winkler (see [28]) which we now introduce. Throughout this section we assume that the initial states are  $X_0 = \eta_{\max}$  and  $Y_0 = \eta_{\min}$ . Given a sequence  $u = (u_1, u_2, \dots)$  of vertices of  $G^*$ , let  $X^u = (X_t^u)_{t \geq 0}$  be the Glauber dynamics such that for every  $t \geq 1$ , the chain is updating the vertices according to the sequence  $u$ , i.e., at time  $t$ ,  $X_t^u$  chooses vertex  $u_t$  to update. Let  $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2, \dots)$  be a sequence of independent and identically distributed uniform random vertices of  $G^*$ . Notice that  $X$ , the (original) Glauber dynamics, satisfies

$$(7.2) \quad X \stackrel{d}{=} X^{\mathbf{U}}.$$

Given a 0/1 sequence  $\gamma = (\gamma_t)_{t \geq 1}$ , and a sequence of vertices  $u = (u_1, u_2, \dots)$ , we define  $X^{u, \gamma} = (X_t^{u, \gamma})_{t \geq 0}$  to be the *censored* version of  $X^u$ , which is restricted, in addition, to changing the configuration at vertex  $u_t$  at time  $t$  only if  $\gamma_t = 1$  (if  $\gamma_t = 0$ , then  $X_t = X_{t-1}$ ).

Given a sequence of vertices  $u$ , we can couple  $(X_t^u, Y_t^u)$  using the same “mechanism” as above. The only difference is that there is no “choice” of vertex for this coupling (as the vertex to be updated is predetermined by  $u$ ). Similarly, we define the coupling  $(X_t^{u, \gamma}, Y_t^{u, \gamma})$  for a sequence of vertices  $u$  and a censoring sequence  $\gamma$ . In all these cases we denote such a coupling by  $\mathcal{L}$ . In particular, similarly to (7.2), we have that

$$(7.3) \quad (X, Y) \stackrel{d}{=} (X^{\mathbf{U}}, Y^{\mathbf{U}}).$$

We will “censor”  $u$  in the following way to ease the calculation of the coupling probability. To couple both copies  $(X_t)$  with  $(Y_t)$  using  $\mathcal{L}$  it is enough to get the root to agree in both copies and then get the leaves to agree. Given a sequence  $u$ , we call a “scan” a subsequence  $u_{i_0}, u_{i_1}, \dots, u_{i_b}$ , where the root is visited and then all the leaves, that is,  $u_{i_0} = r$  and  $\{u_{i_1}, \dots, u_{i_b}\} = \{w_1, \dots, w_b\}$ . We define  $\gamma^u$  as

a 0/1 sequence maximizing the number of nonoverlapping scans in  $u_1, \dots, u_T$ . (If there is more than one such sequence just choose an arbitrary one.) We say that  $\gamma^u$  is a  $k$ -scanning of  $u$  if the sequence  $(u_t)_{t \leq T: \gamma_t^u = 1}$  consists of at least  $k$  scans. Let  $\mathcal{S}_k = \{u : \gamma_u \text{ is a } k\text{-scanning}\}$ , the set of sequences that contain at least  $k$  scans before time  $T$ .

Notice that under the coupling  $\mathcal{L}$ , when the root is unoccupied, to get the leaves to agree it is enough to just choose them. Thus, the coupling probability after one scan is the probability of coupling the root (when chosen and updated), which is at least  $1/(1 + \lambda)$ . Therefore,

$$(7.4) \quad \Pr_{\mathcal{L}} [X_T^{u, \gamma^u} \neq Y_T^{u, \gamma^u}] \leq \left(1 - \frac{1}{\lambda + 1}\right)^k \quad \forall u \in \mathcal{S}_k.$$

Now, to prove (7.1), let  $k = 3(1 + \lambda) \ln(b + 1)$ . We have

$$(7.5) \quad \Pr_{\mathcal{L}} [X_T \neq Y_T] = \sum_u \Pr_{\mathcal{L}} [X_T^u \neq Y_T^u] \Pr_{\mathbf{U}} [u] \\ \leq \Pr_{\mathbf{U}} [u \notin \mathcal{S}_k] + \sum_{u: u \in \mathcal{S}_k} \Pr_{\mathcal{L}} [X_T^u \neq Y_T^u] \Pr_{\mathbf{U}} [u].$$

First we bound  $\Pr_{\mathbf{U}} [u \notin \mathcal{S}_k]$ . Let  $\tau_u$  be the first time  $u$  contains  $k$  consecutive scans. ( $\tau_u$  is a positive random variable which can be equal to  $\infty$ .) By the coupon collector,  $\mathbf{E}_{\mathbf{U}} [\tau_u] = k(b + 1)(1 + \ln b)$ . Using Markov's inequality we have

$$(7.6) \quad \Pr_{\mathbf{U}} [u \notin \mathcal{S}_k] = \Pr_{\mathbf{U}} [\tau_u > 20k(b + 1)(1 + \ln b)] \leq 1/20.$$

Now to bound  $\Pr_{\mathcal{L}} [X_T^u \neq Y_T^u]$  we use the following censoring lemma of Peres and Winkler.

LEMMA 7.3 (see [28]). *For any  $u, \gamma$ , and  $t$ ,*

$$X_t^u \preceq^d X_t^{u, \gamma^u} \text{ and } Y_t^{u, \gamma^u} \preceq^d Y_t^u.$$

Also,

$$(7.7) \quad \|\mu_{X_t^u} - \pi\|_{TV} \leq \|\mu_{X_t^{u, \gamma^u}} - \pi\|_{TV} \text{ and } \|\mu_{Y_t^u} - \pi\|_{TV} \leq \|\mu_{Y_t^{u, \gamma^u}} - \pi\|_{TV}.$$

Notice that the censoring lemma allows us to bound the variation distance, but only starting at the extremal initial configurations. As the extremal configurations are not necessarily the worst case for variation distance, we cannot use the censoring lemma alone. But, as discussed before, the monotonicity of the local coupling allows us to assume extremal initial configurations.

To bound the coupling probability in terms of the coupling probability of the censored chain, we use as an intermediate proxy the variation distance in the following manner:

$$\Pr_{\mathcal{L}} [X_T^u \neq Y_T^u] \\ \leq \sum_{v \in V} \Pr_{\mathcal{L}} [X_T^u(v) \neq Y_T^u(v)] \\ = \sum_{v \in V} \|\mu_{X_T^u(v)} - \mu_{Y_T^u(v)}\|_{TV} \\ \leq (b + 1) \|\mu_{X_T^u} - \mu_{Y_T^u}\|_{TV}$$

$$\begin{aligned}
&\leq (b+1)(\|\mu_{X_T^u} - \pi\|_{\text{TV}} + \|\mu_{Y_T^u} - \pi\|_{\text{TV}}) \\
&\leq (b+1)(\|\mu_{X_T^{u,\gamma_u}} - \pi\|_{\text{TV}} + \|\mu_{Y_T^{u,\gamma_u}} - \pi\|_{\text{TV}}) \quad \text{by (7.7)} \\
&\leq (b+1)(1 - 1/(\lambda+1))^{3(\lambda+1)\ln(b+1)} \quad \text{by (7.4) and the coupling lemma} \\
&\leq 1/(b+1)^2,
\end{aligned}$$

where the first equality follows by the fact that for any monotone coupling of the monotone two-spin system, when projecting on a specific vertex  $v$ , there is actually only one way to couple and hence the probability equals the total variation distance.

Combining with (7.5) and (7.6), we have that

$$\Pr_{\mathcal{L}} [X_T \neq Y_T \mid X_0 = \eta_{\max}, Y_0 = \eta_{\min}] \leq \frac{1}{20} + \frac{1}{(b+1)^2},$$

which implies (7.1) and thus Lemma 7.2 follows.

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#### REFERENCES

- [1] D. ACHLIOPTAS AND A. COJA-OGHLAN, *Algorithmic barriers from phase transitions*, in Proceedings of the 49th Annual IEEE Symposium on Foundations of Computer Science (FOCS), 2008, pp. 793–802.
- [2] D. ALDOUS, *Random walks on finite groups and rapidly mixing Markov chains*, in Séminaire de Probabilités XVII, Lecture Notes in Math., 986, Springer, New York, 1983, pp. 243–297.
- [3] N. BERGER, C. KENYON, E. MOSSEL, AND Y. PERES, *Glauber dynamics on trees and hyperbolic graphs*, Probab. Theory Related Fields, 131 (2005), pp. 311–340.
- [4] N. BHATNAGAR, A. SLY, AND P. TETALI, *Reconstruction threshold for the hardcore model*, in Proceedings of 14th International Workshop, RANDOM 2010, pp. 434–447.
- [5] N. BHATNAGAR, J. VERA, E. VIGODA, AND D. WEITZ, *Reconstruction for colorings on trees*, SIAM J. Discrete Math., 25 (2011), pp. 809–826.
- [6] G. R. BRIGHTWELL AND P. WINKLER, *A second threshold for the hard-core model on a Bethe lattice*, Random Structures Algorithms, 24 (2004), pp. 303–314.
- [7] A. COJA-OGHLAN AND C. EFTHYMIU, *On independent sets in random graphs*, in Proceedings of the 22nd Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), 2011, pp. 136–144.
- [8] J. DING, E. LUBETZKY, AND Y. PERES, *Mixing time of critical Ising model on trees is polynomial in the height*, Comm. Math. Phys., 295 (2010), pp. 161–207.
- [9] C. DASKALAKIS, E. MOSSEL, AND S. ROCH, *Optimal phylogenetic reconstruction*, in Proceedings of the 38th Annual ACM Symposium on Theory of Computing (STOC), 2006, pp. 159–168.
- [10] M. E. DYER, A. SINCLAIR, E. VIGODA, AND D. WEITZ, *Mixing in time and space for lattice spin systems: A combinatorial view*, Random Structures Algorithms, 24 (2004), pp. 461–479.
- [11] A. GALANIS, Q. GE, D. STEFANKOVIC, E. VIGODA, AND L. YANG, *Improved inapproximability results for counting independent sets in the hard-core model*, Proceedings of the 15th International Workshop, RANDOM 2011, 2011, pp. 567–578.
- [12] A. GALANIS, D. STEFANKOVIC, AND E. VIGODA, *Inapproximability of the Partition Function for the Antiferromagnetic Ising and Hard-Core Models*, preprint, arXiv:1203.2226, 2012.
- [13] A. GALANIS, D. STEFANKOVIC, AND E. VIGODA, *Inapproximability for Antiferromagnetic Spin Systems in the Tree Non-Uniqueness Region*, preprint, arXiv:1305.2902, 2013.
- [14] H. O. GEORGI, *Gibbs Measures and Phase Transitions*, de Gruyter Stud. Math., 9, Walter de Gruyter, Berlin, 1988.
- [15] A. GERSCHENFELD AND A. MONTANARI, *Reconstruction for models on random graphs*, in Proceedings of the 48th Annual IEEE Symposium on Foundations of Computer Science (FOCS), 2007, pp. 194–204.
- [16] T. HAYES, J. VERA, AND E. VIGODA, *Randomly coloring planar graphs with fewer colors than the maximum degree*, in Proceedings of the 39th ACM Symposium on Theory of Computing (STOC), 2007, pp. 450–458.

- [17] F. KELLY, *Loss networks*, Ann. Appl. Probab., 1 (1991), pp. 319–378.
- [18] F. KRZAKALA, A. MONTANARI, F. RICCI-TERSENGHI, G. SEMERJIAN, AND L. ZDEBOROVÁ, *Gibbs states and the set of solutions of random constraint satisfaction problems*, Proc. Natl. Acad. Sci. USA, 104 (2007), pp. 10318–10323.
- [19] G. LAWLER AND A. SOKAL, *Bounds on the  $L^2$  spectrum for Markov chains and Markov processes: A generalization of Cheeger’s inequality*, Trans. Amer. Math. Soc., 309 (1988), pp. 557–580.
- [20] D. A. LEVIN, Y. PERES, AND E. L. WILMER, *Markov Chains and Mixing Times*, AMS, Providence, RI, 2009.
- [21] B. LUCIER AND M. MOLLOY, *The Glauber dynamics for colourings of bounded degree trees*, SIAM J. Discrete Math., 25 (2011), pp. 827–853.
- [22] F. MARTINELLI, *Lectures on Glauber Dynamics for Discrete Spin Models*, Lecture Notes in Math. 1717, Springer, New York, 2000.
- [23] F. MARTINELLI, A. SINCLAIR, AND D. WEITZ, *The Ising model on trees: Boundary conditions and mixing time*, Comm. Math. Phys., 250 (2004), pp. 301–334.
- [24] F. MARTINELLI, A. SINCLAIR, AND D. WEITZ, *Fast mixing for independent sets, colorings, and other models on trees*, Random Structures Algorithms, 31 (2007), pp. 134–172.
- [25] A. MONTANARI, R. RESTREPO, AND P. TETALI, *Reconstruction and clustering in random constraint satisfaction problems*, SIAM J. Discrete Math., 25 (2011), pp. 771–808.
- [26] E. MOSSEL, *Survey: Information flow on trees*, in Graphs, Morphisms and Statistical Physics, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., AMS, Providence, RI, 2004, pp. 155–170.
- [27] E. MOSSEL, *Reconstruction on trees: Beating the second eigenvalue*, Ann. Appl. Probab., 11 (2001), pp. 285–300.
- [28] Y. PERES AND P. WINKLER, *Can extra updates delay mixing?* Comm. Math. Phys., 323 (2013), pp. 1007–1016.
- [29] A. SINCLAIR AND M. JERRUM, *Approximate counting, uniform generation and rapidly mixing Markov chains*, Inform. Comput., 82 (1989), pp. 93–133.
- [30] A. SLY, *Reconstruction of random colourings*, Comm. Math. Phys., 288 (2009), pp. 943–961.
- [31] A. SLY, *Computational transition at the uniqueness threshold*, in Proceedings of the 51st Annual IEEE Symposium on Foundations of Computer Science (FOCS), 2010, pp. 287–296.
- [32] A. SLY AND N. SUN, *The computational hardness of counting in two-spin models on  $d$ -regular graphs*, in Proceedings of the 53rd Annual IEEE Symposium on Foundations of Computer Science (FOCS), 2012, pp. 361–369.
- [33] A. D. SOKAL, *A personal list of unsolved problems concerning lattice gases and antiferromagnetic Potts models*, Markov Process. Related Fields, 7 (2001), pp. 21–38.
- [34] P. TETALI, J. C. VERA, E. VIGODA, AND L. YANG, *Phase transition for the mixing time of the Glauber dynamics for coloring regular trees*, in Proceedings of the 21st Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), 2010, pp. 1646–1657.
- [35] D. WEITZ, *Counting independent sets up to the tree threshold*, in Proceedings of the 38th Annual ACM Symposium on Theory of Computing (STOC), 2006, pp. 140–149.