

IMPORTANT: Write each problem on a separate sheet of paper. Write your name on each sheet. (Why? To speed up the grading we will use parallelization—each problem will be graded by a different TA. At the beginning of the class (on the date when the homework is due) there will be a separate pile for each problem.)

Homework 5 is due: Dec. 6 (Thursday); collected before class in Wegmans 1400.

Problem sessions before Homework 5 is due:

Dec. 3 (Monday) 6:15pm - 7:15pm in Hylan 202

Dec. 4 (Tuesday) 6:15pm - 7:15pm in in Goergen 109

Dec. 5 (Wednesday) 6:15pm - 7:15pm in Gavett 310

Dec. 5 (Wednesday) 7:40pm - 8:40pm in Hutchinson 473

Comprehensive Final Exam is on Dec. 20 (Thursday) 4:00pm - 7:00pm in Douglass Ballroom.

1 Homework - solve and turn in

You will need the following concept for the first problem. Let a_0, a_1, \dots, a_{n-1} and b_0, b_1, \dots, b_{n-1} be two sequences of integers. The **convolution** of the two sequences is a sequence $c_0, c_1, \dots, c_{2n-2}$ where

$$c_k = \sum_i a_i b_{k-i}.$$

Thus, for example $c_0 = a_0 b_0, c_1 = a_0 b_1 + a_1 b_0, \dots, c_{2n-2} = a_{n-1} b_{n-1}$. You will need the following fact for the first two problems: the convolution of two sequences can be computed in time $O(n \log n)$.

5.1 (due Dec 6) Let A and B be two sets. Their sum $A + B$ is defined to be

$$A + B := \{a + b \mid a \in A, b \in B\},$$

for example if $A = \{1, 4, 6\}$ and $B = \{2, 5\}$ then $A + B = \{3, 6, 8, 9, 11\}$. Give $O(n \log n)$ algorithm whose input is two sets $A, B \subseteq \{1, \dots, n\}$ and the output is set $A + B$.

5.2 (due Dec 6) We are given a string S over alphabet $\Sigma = \{a, b\}$. We want to find a cyclic rotation of S that disagrees with the original string S on the maximal number of places.

For example, if $S = ababb$ then there are 5 cyclic rotations of S : $ababb$ (0), $babba$ (4), $abbab$ (2), $bbaba$ (2), $babab$ (4); the numbers in the parenthesis give the number of places on which the rotations disagree with S . Thus the output should be either $babba$, or $babab$.

Give an $O(n \log n)$ -time algorithm for the problem.

5.3 (due Dec 6) The Hadamard matrices H_0, H_1, \dots , are defined as follows.

- H_0 is the 1×1 matrix [1].
- For $k > 0$, H_k is the $2^k \times 2^k$ matrix

$$H_k = \left[\begin{array}{c|c} H_{k-1} & H_{k-1} \\ \hline H_{k-1} & -H_{k-1} \end{array} \right].$$

Show that if v is a column vector of length $n = 2^k$ then the matrix-vector product $H_k v$ can be calculated using $O(n \log n)$ operations. (Assume that all the numbers involved are small enough that basic arithmetic operations take $O(1)$ time.)